Non Separable Components and 2-Connected Graphs in Tough Graphs

SAN SAN TINT[1], KHAING KHAING SOE WAI[2]
[1, 2]Department of Engineering Mathematics, Technological University, Myitkyina, Myanmar.

Abstract- In this paper we mention non separable components of longest cycles. And then we establish bounds for the length of a longest cycle C in a 2-connected graph G in terms of the minimum degree δ and the toughness t. It is shown that C is a Hamiltonian cycle or $|C| \geq (t+1)\delta + t$.

Indexed Terms- non separable components, 2-connected graph, induced subgraph, toughness, maximum degree, minimum degree, longest cycle, and neighbourhood.

I. INTRODUCTION

A graph is finite if its vertex set and edge set are finite. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of G. Now suppose that E' is a nonempty subset of E. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E', is called the subgraph of G induced by E' and is denoted by $G[E']$; $G[E']$ is an edge - induced subgraph of G. A vertex-cut in a graph G is a set U of vertices of G such that G − U is disconnected. A complete graph has no vertex-cut.

The vertex-connectivity or simply the connectivity $\kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete , and $\kappa(G) = n - 1$ if $G = K_n$ for some positive integer n . Hence $\kappa(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. If G is either trivial or disconnected, $\kappa(G) = 0$. G is said to be k- connected if $\kappa(G) \geq k$. All non-trivial connected graphs are 1 -connected. If G is non complete graph and t is a nonnegative real number such that $t \leq \frac{|S|}{\omega(G-S)}$ for every vertex - cut S of G , then G is defined to be t -tough. If G is a t - tough graph and s is a nonnegative real number such that s < t, then G is also s-tough. The maximum real number t for which a graph G is a t-tough is called the toughness of G and is denoted by t(G) . A connected graph with at least one cut vertex is called a separable graph, otherwise it is nonseparable.). The degree (or valency) of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges.

We denote by $\delta(G)$ and μ_G the minimum and maximum degrees, respectively of vertices of G. The set of neighbours of a subgraph H of G , denoted N(H), is the set of vertices in V(G) – V(H) adjacent to at least one vertex of H ; d(H) = |N(H)| is the degree of a subgraph H of G

II. NONSEPARABLE COMPONENTS

We still have to investigate non separable components of longest cycles. For an induced subgraph H of G, we set
$X(H) = \{x \in N_{G-H}(H) : |N_H(x,x')| \geq 2 \text{ for each } x' \in N_{G-H}(H) - \{x\}\}$
and $Y(H) = \{y \in V(H) : N_{G-H-X(H)}(y) \neq \varnothing\}$.

Notice that, in fact, $|N_{G-H-X(H)}(y)| \geq 2$ for all y ∈ Y (H).

If V (H) =Y (H), set $\mu_H = \max_{v \in V(H)} d(v)$.
Otherwise, let $\mu_H = |X(H) \cup Y(H)|$.

If $N_{G-H}(H) \neq V(G-H)$ and $V(H) \neq Y(H)$, then $X(H) \cup Y(H)$ is a vertex-cut of G. Anyway, $\mu_H \geq \kappa_G \geq 2t$, where κ_G denotes the connectivity of G.

2.1 Lemma

Let C be a maximal cycle in a graph G, and let H be a non separable component of G – V(C).

If Y (H) = V (H), then $|C| \geq (t+1)\ (\mu_H + |H| - 1) + t$.

Proof

Let X = X (H) and Y = Y (H). By the definition of X and Y, we obtain for each v ∈ V (H) that

$$|N_C(H)| \geq |X| + d_{C-X}(v) + 2|Y| - 2 \geq d(v) + |Y| - 1$$

.Therefore, $|N_C(H)| \geq \mu_H + |H| - 1$ and the results follows , C is a maximal cycle in a graph G of toughness t and let H be a component of G – C since we have $|C| \geq (t+1)|N_C(H)| + t$.

2.2 Lemma

Let C be a longest cycle in a 2-connected graph G, and let H be a component of G – V(C) such that $|H| \leq 2$. Then $|C| \geq (t + 1)\delta + t$.

Proof

If $|H| = 1$, we obtain our result from C is a longest cycle.

Now let V (H) = {v, w}. We fix a cyclic orientation on C and label $N_C(H) = \{x_1. \ldots x_m\}$ accordingly.

Since we have $|C| \geq (t + 1)\ m + t$ and, thus, we are done, if $d_C(v) < m$ or $d_C(w) < m$. For the rest of this proof, let $N_C(v) = N_C(w)$.

Then $|C(x_i, x_{i+1})| \geq 2$ for $1 \leq i \leq m$.

It suffices to show that $|C| \geq (t + 1)\ (m + 1) + t$.

(1)

We consider the vertex-cut $S = V(C) - \{x_1^+, \ldots, x_m^+\}$.

Clearly, there is no C-arc $Q = Q[x_j^+, x_k^+]$ or $Q = Q[x_j^{++}, x_k^+]$ for $1 \leq j < k \leq m$.

If there is no C-arc $Q = Q[x_j^{++}, x_k^{++}]$, then $G - (S - \{x_1^{++}, \ldots, x_m^{++}\})$ has at least m+1components. In this event, $t\ (m + 1) \leq |C| - 2m$.

This yields (1), since $m = \frac{m}{2} + \frac{m}{2} \geq 1 + t$.

Now let us consider a C-arc $Q = Q[x_j^{++}, x_k^{++}]$. This Q gives rise to a cycle C′ with vertex set $V(Q) \cup \{v, w\} \cup V(C) - \{x_j^+, x_k^+\}$.

Hence, $|Q| = 2$ and C′ is a longest cycle.

Let H_j denote the component of $G - C[x_j^{++}, x_j]$ that contains x_j^+. As H_j is a component of G – C′, we are done by a longest cycle if $|H_j| = 1$. Hence, we may assume $|H_j| \geq 2$ and, therefore $N_{G-C}(x_j^+) \neq \emptyset$.

Abbreviate $w_j = x_j^{+++}$ and let H_j' denote the component of $G - C[w_j, x_j]$ that contains H_j. Since C is a longest cycle, we derive that x_j^+ is a cut vertex of H_j'. If $w_j \in N_C(H)$ or there exists a C-arc $Q' = Q'[x_i^+, w_j]$ For some $x_i \in N_C(H) - \{x_j\}$, then there exists a cycle C″ with vertex set $V(C) \cup \{v,w\} - \{x_j^+, x_j^{++}\}$, and H_j' is a component of G – C″. Since C″ must be a longest cycle.

In the remaining case, when $|C(x_j, x_{j+1})| \geq 3$ and there is no C-arc $Q' = Q'[x_i^+, w_j]$ with $x_i \in N_C(H) - \{x_j\}$, we consider the component H_j'' of $G - C(w_j, x_j]$ that contains H_j'. It follows that x_j^+ is a cut vertex also of H_j''. For otherwise there exists a C-arc $Q' = Q'[w_j, x_j^+]$. But Q′ and Q give rise to a cycle C″ with vertex set $V(Q') \cup \{v, w\} \cup V(C) - \{x_k^+\}$, which would be longer than C. Therefore, in fact, $G - (S \cup \{x_j^+\} - \{x_j^{++}, w_j\})$ has at least m +

2 components. Thus, t (m + 2) ≤ ($|S|-2$) + 1 = $|C|$ − m − 1. Equivalently, (1).
This completes the proof of Lemma 2.2.

The next lemma indicates that a 2-connected graph H either has many vertices v such that $d_H(v)$ ≤ D (H) or contains some vertex v such that $d_H(v) \le \frac{D(H)+2}{2}$. It is used to settle the cases addressed in Lemma 2.4.

2.3 Lemma
Let H be a 2-connected graph and let
$Y' = \{v \in V(H): D(H) \ge d_H(v)\}$. If $d_H(v) \ge 2|Y'| - 1$ for each $v \in V(H)$, then $|Y'| \ge 10$.

Proof
Determine a, b ∈ V (H) such that
$D = D(H) = D_H(a, b)$, and let P be a longest (a, b)-path in H. Let r denote the number of components of H − P. Then r > 0, since otherwise
$|Y'| = |H| = D + 1 \ge d_H(v) + 1 \ge 2|Y'|$, which is absurd.
Claim 1. $|Y' - V(P)| \ge 2r$.
Proof of Claim 1
Choose a component L_0 of H − P and label $N_{P[a,b)}(L_0) = \{x_1, \dots, x_m\}$ in order from a to b. If x_i^+ has a neighbor outside P, pick a component L_i of H − P such that $x_i^+ \in N_P(L_i)$. Because P is longest, we have $L_j \ne L_k$ for distinct x_i^+, x_k^+, we obtain
$|Y'| \ge m + \min(|L_0|, 2) \ge 2$.
If $V(L_0) = \{w_0\}$, then $w_0 \in Y'$ and
$|Y'| \ge m + 1 \ge d_H(w_0) \ge 2|Y'| - 1$, contrary to
$|Y'| \ge 2$. Hence, in fact, each component of G − P has at least two vertices and it yields Claim 1.
Claim 2. $|Y' \cap V(P)| \ge \frac{D - r(r-1)}{2}$.
Proof of Claim 2
Let $L_1, \dots, L_r$ be the components of H − P. We "color" the edge vw of P by the pair (i, j),if $v \in N_P(L_i)$ and $w \in N_P(L_j)$. Since P is a longest (a, b)-path, we have i ≠ j, and each color occurs at most once on P. Therefore at least D− r(r − 1) edges are uncolored. Each of those edges has at least one end vertex in V (P) − $N_P(H-P)$ and, consequently, in Y′. Hence Claim 2.

For v ∈ Y′, we have $D \ge d_H(v) \ge 2|Y'| - 1$. We infer by the above claims that $2|Y'| \ge 4r + D - r(r-1) \ge 4r + 2|Y'| - 1 - r(r-1)$.
Hence, 4r ≤ r(r − 1) + 1 and, therefore, r ≥ 5. Claim 1 yields $|Y'| \ge 10$.

2.4 Lemma
Let C be a maximal cycle in a 2-connected graph G Such that c (G) − $|C| \le 2$, and let H be a nonseparable component of G − V(C).
Further, let $Y' = \{v \in V(H): D(H) \ge d_H(v)\}$.
If Y′ ⊆ Y (H) or $|C| \ge (t + 1)(\mu_H + D(H)) + t$, then $|C| \ge (t + 1) d(v) + t$ for some v ∈ V (H).
Proof
We abbreviate Y = Y (H), X = X (H), and
D = D (H). In view of Lemma 3.1.1, we may assume that V (H) ≠ Y, so that X ⋃ Y is a vertex-cut. For v ∈ Y′ − Y, we have $d_H(v) \le D$ and $d_C(v) \le |X|$.
Consequently,
$|C| \ge (t + 1)(\mu_H + D(H)) + t \ge (t + 1) d(v) + t$.
Therefore it remains to consider the case when Y′ ⊆ Y. If $d_H(v) \le 2|Y| - 2$ for some v ∈ V(H), then
$|N_C(H)| \ge d_C(v) + 2|Y| - 2 \ge d_C(v) + d_H(v) \ge d(v)$, and the result follows from Lemma 2.2.3.
Now assume that $d_H(v) \ge 2|Y| - 1$ for all v ∈ V(H). then $|H| \ge 3$, since otherwise for any v ∈ V(H) we would have
$|Y| \ge |Y'| = |H| \ge d_H(v) + 1 \ge 2|Y|$, which is absurd. Therefore, H is 2-connected, and we can apply Lemma 2.1 to obtain $|Y'| \ge 10$.

Fix a cyclic orientation on C. For each $x \in N_C(H)$, let x^* denote the first vertex on C(x, x] in $N_C(H)$. For each y ∈ Y(H), we determine a vertex $\hat{y} \in N_{C-X}(y)$ such that $|N_H(\hat{y}, \hat{y}^*)| \geq 2$.

Let $Y = \{\hat{y} : y \in Y\}$, and abbreviate $\hat{X} = X \cup \hat{Y}$. By construction, we obtain that $|N_H(x, x')| \geq 2$ for any distinct x, x′ ∈ X.

Picking some $y_0 \in Y'$, we label

$$X \cup \left(\hat{Y} - \{\hat{y}_0\}\right) \cup N_{C-X}(y_0) = \{x_1, \ldots, x_m\}$$

According to the fixed orientation.

For $x_i \in X \cup \hat{Y} - \{\hat{y}_0\}$, let u_i, denote the vertex on C such that $|C(x_i, u_i)| = \left\lfloor \frac{D+2-h}{2} \right\rfloor$,

Where h = c (G) – $|C|$.

For the remaining x_i, set $u_i = x_i^{++}$. Since no C-arc joins distinct segments of the form $C(x_i, u_i)$. Therefore, S = V $\left(\bigcup_{i=1}^{m} C[u_i, x_{i+1}]\right)$ is a vertex-cut of G, and G – S has at least $|X \cup Y| - 1 + |N_{C-X}(y_0)| + 1$ components. Hence

$$t(m=1) \leq |S| = |C| - m - \left(|X \cup Y| - 1\right) \left\lfloor \frac{D-h}{2} \right\rfloor$$ And,

Equivalently,

$$|C| \geq (t+1)\, m + t + \left(|X \cup Y| - 1\right) \left\lfloor \frac{D-h}{2} \right\rfloor.$$

(2)

If $|X \cup Y| \geq 2t + 3$, then (2) yields

$$|C| \geq (t+1)\, d_C(y_0) + t + (t+1)\,(D - 1 - h + |Y| - 1)$$
$$\geq (t+1)\,(d(y_0) + 6) + t.$$

In the remaining case, when $|X \cup Y| < 2t + 3$, let us first assume that $D < 2 d_H(v) - 2$ for all v ∈ V (H). Then the graph H has a vertex-cut T = {a, b}. Let L be a component of H – T such that $|V(L) \cap Y|$ is minimum. If $V(L) \subseteq Y$, we pick v ∈ V (L) and obtain $2|Y| - 1 \leq d_H(v) < |Y| - 1 + |T| = |Y| + 1$, contrary to $|Y| \geq |Y'| \geq 10$.

If V(L) – Y ≠ ∅, then contrary to $|Y| \geq |Y'| \geq 10$.

$$2t \leq |X| + |T| + |Y \cap V(L)| \leq |X| + 2 + \frac{|Y|}{2} < 2t + 5 - \frac{|Y|}{2} \leq 2t,$$

Which is absurd. Hence, in fact, $D \geq 2 d_H(v) - 2$ for some v ∈ V (H). As v ∈ Y′, we can apply (2).

From $|X \cup Y| < 2t + 3$, we derive $t > \frac{7}{2}$ and, in particular, $|X \cup Y| - 1 \geq 2t - 1 > t + 2$.

Thus we obtain $|C| \geq (t+1)\, d_C(v) + t + (t+1) \left(|Y| - 1 + d_H(v) - \frac{h+3}{2}\right) > (t+1)\,(d(v) + 6) + t.$

III. CONCLUSION

In this paper we have to investigate nonseparable components of longest cycles. The relation between toughness, minimum degree and the longest cycle is explored. It is shown that C is a Hamiltonian cycle or $|C| \geq (t+1)\delta + t$.

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