Toughness in a Cubic Graph

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Abstract- In this paper we mention vertex-cut and edge-cut. We establish connectivity, edge-connectivity and minimum degree. And then, we discuss toughness t (G) and independence number (β) ***of a graph. Finally the result reveals that the cubic*** $G_m = G_1$ ***with*** $\frac{4}{3}$ ***toughness is obtained.***

Indexed Terms- connectivity, edge-connectivity, t-tough, toughness, k-cube, cubic graph, coloring number, vertex-cut, independence number, minimum degree

I. INTRODUCTION

A graph with n vertices and m edges consists of a vertex set and an edge set where each edge consists of two vertices called its end-vertices. Two vertices u and v of G are said to be connected if there is a -path in G. A graph is said to be connected if every two of its vertices are connected; otherwise it is disconnected. A graph is simple if it has no loops and no parallel edges. The degree of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote by d (G) and D (G) the minimum and maximum degrees, respectively of vertices of G.

Two simple graphs G and H are isomorphic (written if and only if there is a bijection such that if and only if A complete graph G is a simple graph in which every pair of vertices is adjacent. If a complete graph G has n vertices, then it will be denoted by Kn .

The vertex - connectivity or simply the connectivity k(G) of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete , and k(G) = n - 1 if G = Kn for some positive integer n . Hence k (G) is the minimum number of vertices whose removal results in a disconnected or trivial graph. If G is either trivial or disconnected, k (G) = 0. G is said to be k - connected if k (G) ³ k. All non-trivial connected graphs are 1 - connected.

The edge-connectivity k¢(G) of a graph G is the minimum cardinality of an edge-cut of G if G is non-trivial, and k¢(K1) = 0. So k¢ (G) is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Thus k¢(G) = 0 if and only if G is disconnected or trivial; while k¢(G) = 1 if and only if G is connected . A graph G is k - edge - connected, k ³ 1, if k¢ (G) ³ k.

A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y : if |X| = m and |Y| = n, such a graph is denoted by Km,n.

II. CONNECTIVITY WITH TOUGHNESS

A parameter that plays an important role in the study of toughness is the independence number. Two vertices that are not adjacent in a graph G are said to be independent. A set S of vertices is independent if every two vertices of S are independent. The vertex independence number or simply the independence number $\beta(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G. Let F be a graph. A graph G is F-free if G contains no induced subgraph isomorphic of F. A $K_{1,3}$-free graph is also referred to as a claw-free graph. If G is a noncomplete graph and t is a nonnegative real number such that $t \le \frac{|S|}{\omega(G-S)}$ for every vertex-cut S of G, then G is defined to be t-tough. If G is a t-tough graph and s is a nonnegative real number such that $s < t$, then G is also s-tough. The maximum real number t for which a graph G is a t-tough is called the toughness of G and is denoted by t (G). Since complete graphs do not contain vertex-cuts, this definition does not apply to such graphs.

Consequently, we define $t(K_n) = +\infty$ for every positive integer n. certainly, the toughness of a noncomplete graph is a rational number. Also $t(G) = 0$ if and only if G is a disconnected. It follows that if G is a noncomplete graph, then

$$t(G) = \min \frac{|S|}{\omega(G-S)},$$

Where the minimum is taken over all vertex-cuts S of G.

Determining the toughness of a graph usually involves some experimentation. The goal is to find a vertex-cut S that minimizes $\frac{|S|}{\omega(G-S)}$.

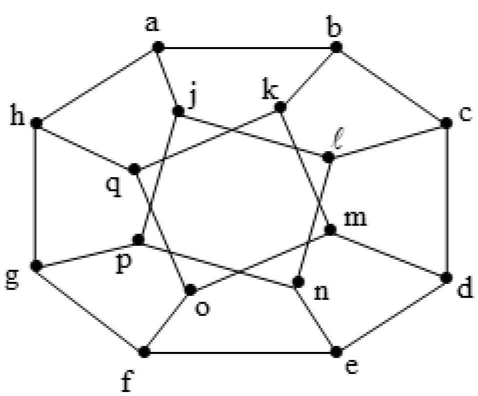

Figure 1 A graph G of toughness $\frac{6}{5}$.

For the graph G of figure 1, $S_1 = \{g, h, c, d, j, k, o, n\}$, $S_2 = \{a, c, e, g, q, m\}$, $S_3 = \{a, d, f, h, l, m, q, p\}$.

$$\frac{|S_1|}{\omega(G-S_1)} = \frac{8}{6}, \quad \frac{|S_2|}{\omega(G-S_2)} = \frac{6}{5}, \quad \frac{|S_3|}{\omega(G-S_3)} = \frac{8}{5}.$$

2.1Theorem

Let G be a connected graph of order $n \geq 3$ that is not complete. For each edge-cut X of G, there is a vertex-cut U of G such that $|U| \leq |X|$.

Proof:

Assume, without loss of generality, that X is a minimum edge-cut of G. Then $G - X$ is a disconnected graph containing exactly two components G_1 and G_2, where G_i has order n_i $(i = 1, 2)$. Thus $n_1 + n_2 = n$. We consider two cases.

Case 1.

Every vertex of G_1 is adjacent to every vertex of G_2. Then $|X| = n_1 n_2$. Since $(n_1 - 1)(n_2 - 1) \geq 0$, it follows that $n_1 n_2 \geq n_1 + n_2 - 1 = n - 1$ and so $|X| \geq n - 1$. Since G is not complete, G contains two nonadjacent vertices u and v. Then $U = V(G) - \{u, v\}$ is a vertex-cut of cardinality $n - 2$ and $|U| < |X|$.

Case 2.

There are vertices u in G_1 and v in G_2 that are not adjacent in G. For each edge e in X, we select a vertex for U in the following way. If u is incident with e, then choose the other vertex (in G_2) incident with e for U; otherwise, select for U the vertex that is incident with e and belongs to G_1. Now $|U| \leq |X|$. Furthermore, $u, v \in V(G - U)$, but $G - U$ contains (u, v)-path, so U is a vertex-cut.

2.2 Theorem

For every graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Proof:

If G is trivial or disconnected, then $\kappa(G) = \kappa'(G) = 0$; so we can assume that G is a nontrivial connected graph. Let v be a vertex of G such that deg $v = \delta(G)$. the removal of the $\delta(G)$ edges of G incident with v results in a graph G' in which v is isolated, so G' is either disconnected or trivial. Therefore, $\kappa'(G) \leq \delta(G)$.

We now verify the other inequality. If $G = K_n$ for some positive integer n, then $\kappa(K_n) = \kappa'(K_n) = n - 1$. suppose next that G is not complete, and let X be an edge-cut such that $|X| = \kappa'(G)$.

By Theorem 2.1 there exists a vertex-cut U such that $|U| \leq |X|$. Thus $\kappa(G) \leq |U| \leq |X| = \kappa'(G)$.

2.3 Corollary

Let G be a graph with vertices $x_1, x_2, ..., x_n, d(x_1) \leq d(x_2) \leq ... \leq d(x_n)$. suppose for some k, $0 \le k \le n$, that $d(x_j) \ge j+k-1$, for $j = 1, 2, ..., n-1-d(x_{n-k+1})$, then G is k-connected.

Proof

Suppose that G is not k-connected. Then there exist $V_1, V_2 \subset V(G)$ such that $V_1 \cap V_2 = \varnothing$, $|V_1| = n_1$, $|V_2| = n_2$, $n_1 + n_2 = n - k + 1$ and $d(x) \le n_i + k - 2$ for $x \in V_i$. Now, $X = \{x_j \mid j \ge n-k+1\}$ is a set of k elements all with a degree larger than or equal to $d(x_{n-k+1})$. Hence, there is at least one $x \in X \cap (V_1 \cup V_2)$. Without loss of generality, say in $x \cap V_2$. Thus $n_2 \ge d(x_{n-k+1}) + 1 - (k-1) = d(x_{n-k+1}) - k + 2$ and $n_1 = n - k + 1 - n_2 \le n - 1 - d(x_{n-k+1})$. Take $x_j \in V_1$ such that j is maximal $(j \ge n_1)$, then $n_1 + k - 1 \le d(x_{n_1}) \le d(x_j) \le n_1 + k - 2$.

Thus, if G is a graph with vertices $x_1, x_2, ..., x_n$, with $d(x_1) \le d(x_2) \le ... \le d(x_n) = \Delta(G)$ and $d(x_j) \ge j$ for $j = 1, 2, ..., n - \Delta(G) - 1$, then G is connected. The reverse is, obviously, not true.

2.4 Corollary

Let $G \ne K_n$ be a graph of order n, then $\kappa(G) \ge 2\delta(G) + 2 - n$.

Proof:

Let $k = 2\delta(G) + 2 - n$. It suffices to show $d(x_j) \ge j+k-1$, for $j = 1, ..., n-1-\delta(G)$ (because $d(x_{n-k+1}) \ge \delta(G)$). This is certainly true if $d(x_j) \ge n - 1 - \delta(G) + k - 1$ for all $j = 1, ..., n-1-\delta(G)$ and $n - 1 - \delta(G) + k - 1 = \delta(G)$.

2.5 Theorem

For every noncomplete graph G,

$$\frac{\kappa(G)}{\beta(G)} \le t(G) \le \frac{\kappa(G)}{2}.$$

Proof:

The independence number is related to toughness in the sense that among all the vertex-cuts S of the noncomplete graph G, the maximum value of $\omega(G-S)$ is $\beta(G)$, so for every vertex-cut S of G, we have that $\kappa(G) \le |S|$ and $\omega(G-S) \le \beta(G)$.

By the definition of $t(G)$,

$$t(G) = \min \frac{|S|}{\omega(G-S)} \ge \frac{\kappa(G)}{\beta(G)}.$$

Let S' be a vertex-cut with $|S'| = \kappa(G)$.

Thus $\omega(G - S') \ge 2$, so

$$t(G) = \min \frac{|S|}{\omega(G-S)} \le \frac{|S'|}{\omega(G-S')} \le \frac{\kappa(G)}{2}.$$

2.6 Theorem [3]

A graph G of order $n \ge 2$ is k-connected $(1 \le k \le n-1)$ if and only if for each pair u, v of distinct vertices there are at least k internally-disjoint (u, v) – paths in G.

2.7 Theorem

If G is a noncomplete claw-free graph, then $t(G) = \frac{1}{2}\kappa(G)$.

Proof:

If G is disconnected, then $t(G) = \kappa(G) = 0$ and the result follows. So we assume that $\kappa(G) = r \ge 1$. Let S be a vertex-cut such that $t(G) = \frac{|S|}{\omega(G-S)}$. suppose that $\omega(G-S) = k$ and that $G_1, G_2, ..., G_k$ are the components of $G-S$.

Let $u_i \in V(G_i)$ and $u_j \in V(G_j)$, where $i \ne j$. Since G is r-connected, it follows by Theorem 2.6 that G contains at least r internally disjoint (u_i, u_j)-paths. Each of these paths contains a vertex of S. Consequently, there are at least r edges joining the vertices of S and the vertices of G_i for each i $(1 \le i \le k)$ such that no two of these edges are incident with the same vertex of S.

Hence there is a set X containing at least kr edges between S and $G-S$ such that any two edges incident with a vertex of S are incident with vertices in distinct components of $G-S$. However, since G is claw-free, no vertex of S is joined to vertices in three components of $G-S$. Therefore,

$$kr = |X| \le 2|S| = 2k\, t(G).$$

So $kr \le 2k\, t(G).$

Thus $t(G) \ge \frac{r}{2} = \frac{1}{2}\kappa(G).$

By Theorem 2.5, $t(G) = \frac{1}{2}\kappa(G).$

III. TOUGHNESS AND INDEPENDENCE NUMBER

In this paper we derive upper bounds on the toughness of cubic graphs in terms of the independence number and coloring parameters.

A coloring of a graph G we mean an assignment of colors to the vertices of G such that any two vertices joined by an edge receive different colors. A color class A is minimal if every vertex of A has a neighbor of every other color.

We show an example of a graph coloring, thus need three colors to color.

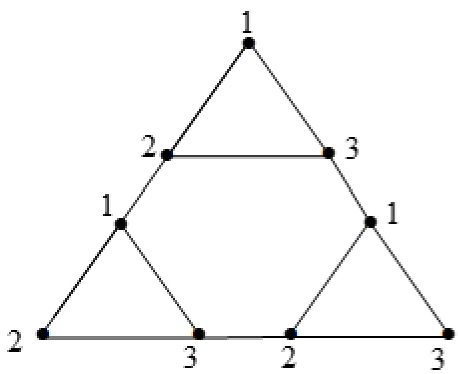

Figure 2. A graph coloring.

A graph is a k-colorable if there is a vertex coloring with k colors. Let be a graph. The chromatic number of G, written is the minimum integer k such that G is k-colorable.

The k-cube is the graph whose vertices are the ordered k-tuples of o's and 1's, two vertices begin joined if and only if they differ in exactly one coordinate.

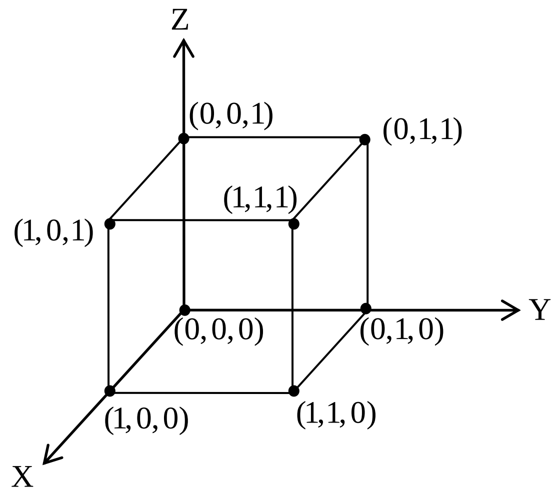

Figure 3. 3-cube.

3.1 Theorem

The k-cube has 2^k vertices, $k\,2^{k-1}$ edges and is bipartite.

Proof:

Each coordinate of a k-tuple can be chosen in two ways, 0 (or) 1.

Therefore there are k-places which has 2^k different ways.

So k-cube has 2^k vertices. Let V be the vertex set of k-cube.Let $v \in V$. Since v is k-tuple, there are k-tuple which differ from v in exactly one coordinate.

There are k-edges between v and these vertices.

Let E_v be the set of edges incident with v.

$$|E_v| = k.$$

For all $v \in V$, we have the sum

$$\sum_{v \in V} |E_v| = k(2^k)$$

But in this sum edge is counted twice.

Therefore 2 (number of edges in k-cube) $= k\,(2^k)$.

We have seen thatthe number of edges in k-cube $= \frac{k2^k}{2} = k2^{k-1}$.

Let $X = \{v \in V \,|\, \text{the number of is in v is odd}\}$.

Let $Y = \{v \in V \,|\, \text{the number of is in v is even}\}$.

Take any edge uv in k-tuple.

Assume that $u \in X$.

Since u and v differ in exactly one coordinate, $v \in Y$. Thus for every edge uv in the k-cube, one end in X and the other end in Y.
Hence k-cube is bipartite.

3.2 Lemma [5]
Let G be a cubic graph with a 3-coloring (A, B, C) such that A is minimal. If $|A| = a, |B| = b,$ and $|C| = c$ then $t(G) \le \frac{3b + a - c}{2b}$.

3.3 Theorem
For a noncomplete cubic graph G on n vertices and independence number β :

$$t(G) \le \min\left(\frac{2n - 3\beta}{n - \beta}, \frac{2\beta}{4\beta - n}\right).$$

Proof:
By the definition of a graph coloring, a graph of maximum degree r has an r-coloring where one of the color classes is a maximum independent set. Let (A, B, C) be a 3-coloring of G where C is a maximum independent set, and subject to this, A is as small as possible. So $|C| = \beta$ and $b \ge \frac{(n - \beta)}{2}$.
Also $3b \ge 3c - a = 3\beta - (n - b - \beta)$ by inequality, whence $b \ge 2\beta - \frac{n}{2}$.
So by the above Lemma 3.2,

$$t(G) \le \frac{3b + a - c}{2b} = 1 + \frac{n - 2\beta}{2b} \le \min\left(\frac{2n - 3\beta}{n - \beta}, \frac{2\beta}{4\beta - n}\right),$$

as required.

This is best possible. For $\beta = \frac{n}{2}$ the theorem gives an upper bound of 1, and any 3-connected cubic bipartite graph has toughness 1. At the other extreme the theorem shows that $t(G) = \frac{3}{2}$ in noncomplete cubic graphs requires that $\beta = \frac{n}{3}$ and thus n a multiple of 3.

Consider also the following cubic graph G_m for m a positive integer. Start with a set $U = \{u_1, \ldots, u_{6m}\}$ of 6m vertices which form a cycle. Then add a set $V = \{v_1, \ldots, v_{3m}\}$ 3m vertices, and connect v_i to u_{2i-1} and u_{2i}. Finally add a set $W = \{w_1, \ldots, w_m\}$ of m vertices, and join w_i to v_{3i-2}, v_{3i} and v_{3i+2}. The graph is illustrated in Figure 1. The graph G_m has $n = 10m$ vertices, toughness $\frac{4}{3}$ and independence number $\beta = \frac{2n}{5}$.
The graph is illustrate for the cubic graph $G_m = G_1$.

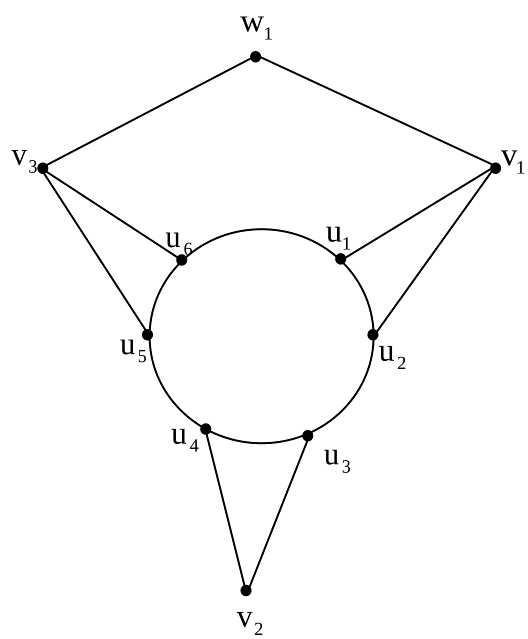

Figure 4. The cubic graph

3.4 Proposition [5]
For a noncomplete graph G, $\frac{\kappa(G)}{\Delta(G)} \le t(G) \le \frac{\kappa(G)}{2}$.

3.5 Lemma
For $m \ge 1$, G_m has toughness $\frac{4}{3}$.

Proof:
Let S be a vertex-cut and suppose $G_m - S$ has k components. Note that by Proposition 3.4, $k \le |S|$ (since G_m is 3-connected). Thus as $\frac{(|S| + 1)}{(k + 1)} \le \frac{|S|}{k}$ we may assume that $G_m - S$ has no vertex-cut.
We show first that we may assume that $S \cap V = \varnothing$. suppose for some w_i some of its neighbors are in S.

Then in $G_m - S$ reinsert the neighbors of w_i and remove w_i instead. This action cannot join two components (since v_j's two neighbors in U are already adjacent). The only way this action can reduce the number of components is if w_i a singleton component in was $G_m - S$. But that means all of w_i's neighbours are in S and since $\frac{(|S|-2)}{(k-1)} < \frac{|S|}{k}$ we are better off, a contradiction. So this action does not decrease the number of components and does not increase the number of vertices removed. Hence we may assume that $S \cap V = \varnothing$.

Let w denote $|S \cap W|$. Since $G_m - S$ has no cut-vertex, for any w_i not in S the vertices $u_{6i-4}, u_{6i-3}, \ldots, u_{6i+3}$ must all lie in the component with w_i. Denote the subpath $u_{6i-4}, u_{6i-3}, \ldots, u_{6i+3}$ by P_i. Now it is not hard to see that the best strategy, once $S \cap W$ is determined, is to remove every alternate vertex of U that lies outside the P_i corresponding to the $w_i \notin S$. The number u of vertices of U removed is equal to the number of components that remain. Also $u \le 3w$. Hence $t \ge \frac{(w+3w)}{(3w)} = \frac{4}{3}$.

IV. CONCLUSION

We conclude that connected graph have connectivity and edge- connectivity. The bounds of connectivity of a graph G are expressed as in terms of minimum degree of G and numbers of vertices in G. The toughness t (G) is discussed which is related to the connectivity and independence number in G. And then the result of the cubic graph G_m has 4/3 toughness.

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