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# The Semi-Implicit Scheme and Its Convergence for The Parabolic Problem

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***Abstract -- In this paper, we consider the parabolic initial boundary value problem. Then, we describe the bilinear form of the parabolic advection- diffusion problem. . Finally, we prove the convergence of the semi implicit scheme for the problem.***

***Indexed Terms - Advection- Diffusion Equation, semi implicit scheme, convergence***

## I. INTRODUCTION

We consider the parabolic advection-diffusion problem as follows:

Given $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$,

find $u \in L^2(0,T;V) \cap C^0([0,T];L^2(\Omega))$ such that

$$\left.\begin{aligned} \frac{d}{dt}(u(t),v) + a(u(t),v) &= (f(t),v),\ \forall v \in V \\ u(0) &= u_0, \end{aligned}\right\} \quad (1)$$

where $V = H_0^1(\Omega)$.

The bilinear form is

$$a(w,v) = \int_\Omega \left[ \varepsilon \nabla w \cdot \nabla v - \frac{1}{2}\sum_{i=1}^{n}(b_i w D_i v - b_i v D_i w) + \mu w v \right] dx. \quad (2)$$

We will prove that $a(\cdot,\cdot)$ is continuous as follows:

$|a(w,v)| =$

$$\left| \int_\Omega \varepsilon \nabla w \cdot \nabla v dx - \frac{1}{2}\int_\Omega \sum_{i=1}^{n}(b_i w D_i v - b_i v D_i w)\, dx + \int_\Omega \mu\, w\, v\, dx \right|$$

$$\leq \varepsilon \int_\Omega |\nabla w||\nabla v| dx + \frac{1}{2}\int_\Omega \|b\|_{L^\infty(\Omega)} |w||\nabla v| dx$$

$$+ \frac{1}{2}\int_\Omega \|b\|_{L^\infty(\Omega)} |v||\nabla w| dx \ + \mu_1 \int_\Omega |w||v| dx.$$

By using Hölder's inequality and Poincaré's inequality, $|a(w,v)|$

$$\leq \varepsilon \|\nabla w\|_0 \|\nabla v\|_0 + \frac{1}{2} C_1 \|\nabla w\|_0 \|\nabla v\|_0$$

$$+ \frac{1}{2} C_2 \|\nabla v\|_0 \|\nabla w\|_0 + \mu_1 C_1 C_2 \|\nabla w\|_0 \|\nabla v\|_0$$

$$|a(w,v)| = (\varepsilon + 1 + \mu_1)(\|\nabla w\|_0^2)^{\frac{1}{2}} (\|\nabla v\|_0^2)^{\frac{1}{2}},$$

where $C_1 C_2 = 1$.

$|a(w,v)| \leq \gamma \|w\|_1 \|v\|_1$, where $\gamma = \varepsilon + 1 + \mu_1$.

Therefore the bilinear form $a(\cdot,\cdot)$ is continuous.

The stability condition for the $\theta$-scheme is,

$$\Delta t \leq \frac{Ch^2 \alpha}{(1-2\theta)\gamma^2}. \quad (2.5)$$

For a small $\varepsilon$ this becomes

$$\Delta t \leq \frac{Ch^2 \varepsilon}{(1-2\theta)(\varepsilon^2 + 1 + \mu_1^2)}.$$

## II. NUMERICAL SCHEMES

We consider the semi-discrete (continuous in time) approximation of the advection-diffusion initial boundary value problem (1):

$$\left.\begin{aligned} &\frac{d}{dt}(u_h(t), v_h) + a(u_h(t), v_h) = (f(t), v_h), \\ &\forall v_h \in V_h,\ t \in (0,T) \\ &\qquad u_h(0) = u_{0,h}. \end{aligned}\right\} (3)$$

Here $V_h \subset H_0^1(\Omega)$ is a suitable finite-dimensional space and $u_{0,h} \in V_h$ approximates the initial datum $u_0$.

2.1) A Semi -Implicit Scheme

We consider the semi-implicit time-discretization approach. This consists in evaluating the principal part of operator L at time level $t_{n+1}$, whereas the remaining parts are considered at the time $t_n$. This scheme reads

for each $n = 0,1,\dots,N-1.$

$$\left.\begin{aligned} &\frac{1}{\Delta t}\left(u_h^{n+1}-u_h^n, v_h\right)+\varepsilon\left(\nabla u_h^{n+1}, \nabla v_h\right)-\sum_{i=1}^{n}\left(b_i u_h^n, D_i v_h\right) \\ &\quad+\left(a_0 u_h^n, v_h\right)=\left(f\left(t_{n+1}\right), v_h\right),\ \forall v_h \in V_h, \\ &u_h^0 = u_{0,h,} \end{aligned}\right\} (4)$$

We study semi-discrete methods based on the so-called θ-scheme:

$$\left.\begin{aligned} &\frac{1}{\Delta t}\left(u_h^{n+1}-u_h^n, v_h\right)+a\left(\theta u_h^{n+1}+(1-\theta)u_h^n, v_h\right)=\left(\theta f\left(t_{n+1}\right)+(1-\theta)f\left(t_n\right), v_h\right), \forall v_h \in V_h, \\ &u_h^0 = u_{0,h}, \end{aligned}\right\}$$

for each $n = 0, 1, \dots, N-1$, where $0 \le \theta \le 1$, $\Delta t = \frac{T}{N}$ is the time step, N is a positive integer, and $u_{0,h} \in V_h$ is a suitable approximation of the initial datum $u_0$.

This includes the schemes: forward Euler $(\theta = 0)$, backward Euler $(\theta = 1)$ and Crank-Nicolson $\left(\theta = \frac{1}{2}\right)$.

The elliptic projection operator $\prod_{1,h}^k$ is defined as follows: For each $v \in V$

$$\prod_{1,h}^k(v) \in V_h \text{ implies } a\left(\prod_{1,h}^k(v), v_h\right) = a\left(v, v_h\right)\ \forall v_h \in V_h.$$

2.2 Theorem

Assume that $u_0 \in H_0^1(\Omega)$ and that the solution u to (1) is such that $\frac{\partial u}{\partial t} \in L^2\left(0,T;H_0^1(\Omega)\right)$ and $\frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right)$. Then $u_h^n$ defined in (4) satisfies for each $n = 0,1,\dots,N$

$$\begin{aligned} \left\|u_h^n - u(t_n)\right\|_0 &\le \left\|\left(I-\prod_{1,h}^k\right)u(t_n)\right\|_0 + \exp\left(C_\varepsilon t_n\right) \\ &\times\left\{\left\|u_{0,h}-\prod_{1,h}^k(u_0)\right\|_0^2 + C_\varepsilon\int_0^{t_n}\left\|\left(I-\prod_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_0^2 ds \right. \\ &\left. + C_\varepsilon(\Delta t)^2\int_0^{t_n}\left(\left\|\frac{\partial u}{\partial t}(s)\right\|_1^2 + \left\|\frac{\partial^2 u}{\partial t^2}(s)\right\|_0^2\right)ds\right\}^{\frac{1}{2}} \end{aligned} \quad (5)$$

where $\prod_{1,h}^k$ is the elliptic projection operator and $C_\varepsilon > 0$ is a non-decreasing function of $\varepsilon^{-1}$.

Proof

Set $\eta_h^n = u_h^n - \prod_{1,h}^k\left(u(t_n)\right)$ in (4).

Then, we obtain

$$\frac{1}{\Delta t}\left(\eta_h^{n+1}+\prod_{1,h}^k\left(u(t_{n+1})\right)-\eta_h^n-\prod_{1,h}^k\left(u(t_n)\right), v_h\right)$$

$$+\varepsilon\left(\nabla\eta_h^{n+1}+\nabla\prod_{1,h}^k\left(u(t_{n+1})\right), \nabla v_h\right)$$
$$-\sum_{i=1}^{n}\left(b_i\eta_h^n + b_i\prod_{1,h}^k\left(u(t_n)\right), D_i v_h\right)$$
$$+\left(a_0\eta_h^n + a_0\prod_{1,h}^k\left(u(t_n)\right), v_h\right) = \left(f\left(t_{n+1}\right), v_h\right).$$

That is,

$$\begin{aligned} &\frac{1}{\Delta t}\left(\eta_h^{n+1}-\eta_h^n, v_h\right)+\varepsilon\left(\nabla\eta_h^{n+1}, \nabla v_h\right)-\sum_{i=1}^{n}\left(b_i\eta_h^n, D_i v_h\right)+\left(a_0\eta_h^n, v_h\right) \\ &= \left(f\left(t_{n+1}\right), v_h\right)-\frac{1}{\Delta t}\left(\prod_{1,h}^k\left(u(t_{n+1})-u(t_n)\right), v_h\right) \\ &-\varepsilon\left(\nabla\prod_{1,h}^k\left(u(t_{n+1})\right), \nabla v_h\right) \end{aligned}$$

$$+\sum_{i=1}^{n}\left(b_i\prod_{1,h}^k\left(u(t_n)\right), D_i v_h\right)-\left(a_0\prod_{1,h}^k\left(u(t_n)\right), v_h\right),$$

$$\frac{1}{\Delta t}\left(\eta_h^{n+1}-\eta_h^n, v_h\right)+\varepsilon\left(\nabla\eta_h^{n+1}, \nabla v_h\right)$$
$$-\sum_{i=1}^{n}\left(b_i\eta_h^n, D_i v_h\right)+\left(a_0\eta_h^n, v_h\right)$$

$$= \left(f\left(t_{n+1}\right), v_h\right) - \frac{1}{\Delta t}\left(\Pi_{1,h}^k\left(u\left(t_{n+1}\right) - u\left(t_n\right)\right), v_h\right)$$
$$-\varepsilon\left(\nabla u\left(t_{n+1}\right), \nabla v_h\right)$$
$$+\sum_{i=1}^n\left(b_i u\left(t_n\right), D_i v_h\right) - \left(a_0 u\left(t_n\right), v_h\right) - \varepsilon\left(\nabla u\left(t_n\right), \nabla v_h\right)$$
$$+\varepsilon\left(\nabla \Pi_{1,h}^k\left(u\left(t_n\right)\right), \nabla v_h\right) + \sum_{i=1}^n\left(b_i u\left(t_{n+1}\right), D_i v_h\right)$$
$$-\sum_{i=1}^n\left(b_i \Pi_{1,h}^k\left(u\left(t_{n+1}\right)\right), D_i v_h\right) - \left(a_0 u\left(t_{n+1}\right), v_h\right)$$
$$+\left(a_0 \Pi_{1,h}^k\left(u\left(t_{n+1}\right)\right), v_h\right). \quad (6)$$

On the other hand, the solution u satisfies

$$\left(\frac{\partial u}{\partial t}\left(t_{n+1}\right), v_h\right) = \left(f\left(t_{n+1}\right) v_h\right) - \varepsilon\left(\nabla u\left(t_{n+1}\right), \nabla v_h\right)$$
$$+\sum_{i=1}^n\left(b_i u\left(t_{n+1}\right), D_i v_h\right) - \left(a_0 u\left(t_{n+1}\right), v_h\right).$$

Thus, ( 6 ) becomes

$$\frac{1}{\Delta t}\left(\eta_h^{n+1} - \eta_h^n, v_h\right) + \varepsilon\left(\nabla \eta_h^{n+1}, \nabla v_h\right) - \sum_{i=1}^n\left(b_i \eta_h^n, D_i v_h\right) + \left(a_0 \eta_h^n, v_h\right)$$

$$= \left(\frac{\partial u}{\partial t}\left(t_{n+1}\right), v_h\right) - \frac{1}{\Delta t}\left(\Pi_{1,h}^k\left(u\left(t_{n+1}\right) - u\left(t_n\right)\right), v_h\right) + \sum_{i=1}^n\left(b_i u\left(t_n\right), D_i v_h\right)$$

$$-\left(a_0 u\left(t_n\right), v_h\right) - \varepsilon\left(\nabla u\left(t_n\right), \nabla v_h\right) + \varepsilon\left(\nabla \Pi_{1,h}^k\left(u\left(t_n\right)\right), \nabla v_h\right)$$

$$-\sum_{i=1}^n\left(b_i \Pi_{1,h}^k\left(u\left(t_{n+1}\right)\right), D_i v_h\right) + \left(a_0 \Pi_{1,h}^k\left(u\left(t_{n+1}\right)\right), v_h\right).$$

Let us further recall that the definition of $\Pi_{1,h}^k$ reads

$$-\varepsilon\left(\nabla u\left(t_n\right), \nabla v_h\right)\big) + \sum_{i=1}^n\left(b_i u\left(t_n\right), D_i v_h\right) - \left(a_0 u\left(t_n\right), v_h\right)$$
$$= -\varepsilon\left(\nabla \Pi_{1,h}^k\left(u\left(t_n\right)\right), \nabla v_h\right) + \sum_{i=1}^n\left(b_i \Pi_{1,h}^k\left(u\left(t_n\right)\right), D_i v_h\right) - \left(a_0 \Pi_{1,h}^k\left(u\left(t_n\right)\right), v_h\right).$$

Since $\frac{\partial}{\partial t}$ commutes with $\Pi_{1,h}^k$ ( 6) can be rewritten as

$$\frac{1}{\Delta t}\left(\eta_h^{n+1} - \eta_h^n, v_h\right) + \varepsilon\left(\nabla \eta_h^{n+1}, \nabla v_h\right) - \sum_{i=1}^n\left(b_i \eta_h^n, D_i v_h\right) + \left(a_0 \eta_h^n, v_h\right)$$

$$= \left(\frac{\partial u}{\partial t}\left(t_{n+1}\right), v_h\right) - \frac{1}{\Delta t}\left(\Pi_{1,h}^k\left(u\left(t_{n+1}\right) - u\left(t_n\right)\right), v_h\right)$$
$$+\sum_{i=1}^n\left(b_i \Pi_{1,h}^k\left(u\left(t_n\right)\right), D_i v_h\right)$$
$$-\left(a_0 \Pi_{1,h}^k\left(u\left(t_n\right)\right), v_h\right) - \varepsilon\left(\nabla \Pi_{1,h}^k\left(u\left(t_n\right)\right), \nabla v_h\right)$$
$$+\varepsilon\left(\nabla \Pi_{1,h}^k\left(u\left(t_n\right)\right), \nabla v_h\right)$$
$$-\sum_{i=1}^n\left(b_i \Pi_{1,h}^k\left(u\left(t_{n+1}\right)\right), D_i v_h\right) + \left(a_0 \Pi_{1,h}^k\left(u\left(t_{n+1}\right)\right), v_h\right),$$

$$\frac{1}{\Delta t}\left(\eta_h^{n+1} - \eta_h^n, v_h\right) + \varepsilon\left(\nabla \eta_h^{n+1}, \nabla v_h\right)$$
$$-\sum_{i=1}^n\left(b_i \eta_h^n, D_i v_h\right) + \left(a_0 \eta_h^n, v_h\right) = \left(\varepsilon_h^n, v_h\right), (7)$$

where $\varepsilon_h^n \in V_h$ is defined by the relation

$$\left(\varepsilon_h^n, V_h\right) = \left(\frac{\partial u}{\partial t}\left(t_{n+1}\right) - \frac{u\left(t_{n+1}\right) - u\left(t_n\right)}{\Delta t}, v_h\right)$$
$$+\frac{1}{\Delta t}\left(\int_{t_n}^{t_n+1}\left(I - \Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\,ds, v_h\right)$$
$$+\sum_{i=1}^n\left(b_i \Pi_{1,h}^k\left(u\left(t_n\right) - u\left(t_{n+1}\right)\right), D_i v_h\right)$$
$$-\left(a_0 \Pi_{1,h}^k\left(u\left(t_n\right) - u\left(t_{n+1}\right)\right), v_h\right). \quad (8)$$

Therefore $\eta_h^n$ satisfies a scheme like ( 4 ). we have

$$\left\|\eta_h^n\right\|_0^2 \le \left(\left\|u_{0,h} - \Pi_{1,h}^k\left(u_0\right)\right\|_0^2 + \frac{C_1}{\varepsilon}\Delta t\sum_{n=0}^{m-1}\left\|\varepsilon_h^n\right\|_{-1,h}^2\right) (9)$$
$$\times \exp\left[C_1\frac{t_n}{\varepsilon}\left(1 + \left\|a_0\right\|_{L^\infty(\Omega)}^2\right)\right].$$

The first term at the right hand side of (8) can be estimated as follows:

$$\left(\frac{\partial u}{\partial t}\left(t_{n+1}\right) - \frac{u\left(t_{n+1}\right) - u\left(t_n\right)}{\Delta t}, v_h\right) = \frac{1}{\Delta t}\left(\int_{t_n}^{t_{n+1}}\left(s - t_n\right)\frac{\partial^2 u}{\partial t^2}(s)\,ds,\ v_h\right)$$
$$= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(s - t_n\right)\left(\frac{\partial^2 u}{\partial t^2}(s),\ v_h\right)ds$$
$$\le \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(s - t_n\right)\left\|\frac{\partial^2 u}{\partial t^2}(s)\right\|_{-1,h}\left\|v_h\right\|_1 ds.$$

The second term at the right hand side of (8 ) can be estimated as follows:

$$\frac{1}{\Delta t}\left(\int_{t_n}^{t_{n+1}}\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)ds,\ v_h\right)\le\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_{-1,h}\|v_h\|_1\,ds.$$

Moreover, we have

$$\sum_{i=1}^{n}\left(b_i\,\Pi_{1,h}^k\left(u(t_n)-u(t_{n+1})\right),D_i v_h\right)\le\left\|\Pi_{1,h}^k\left(u(t_n)-u(t_{n+1})\right)\right\|_{-1,h}\|v_h\|_1$$
$$\le\left\|\Pi_{1,h}^k\left(u(t_n)-u(t_{n+1})\right)\right\|_0\|v_h\|_1$$

and

$$\left(a_0\,\Pi_{1,h}^k\left(u(t_n)-u(t_{n+1})\right),v_h\right)$$
$$\le\|a_0\|_{L^\infty(\Omega)}\left\|\Pi_{1,h}^k\left(u(t_n)-u(t_{n+1})\right)\right\|_{-1,h}\|v_h\|_1$$
$$\le\|a_0\|_{L^\infty(\Omega)}\left\|\Pi_{1,h}^k\left(u(t_n)-u(t_{n+1})\right)\right\|_0\|v_h\|_1.$$

Therefore, (8) becomes

$$\left(\varepsilon_h^n,v_h\right)\le\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(s-t_n)\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_{-1,h}\|v_h\|_1\,ds$$
$$+\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_{-1,h}\|v_h\|_1\,ds$$
$$+\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0\|v_h\|_1+\|a_0\|_{L^\infty(\Omega)}\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0\|v_h\|_1,$$

$$\frac{\left(\varepsilon_h^n,v_h\right)}{\|v_h\|_1}\le\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(s-t_n)\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_{-1,h}dx+\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_{-1,h}ds$$
$$+\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0+\|a_0\|_{L^\infty(\Omega)}\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0.$$

By using inequality, we have

$$\|\varepsilon_h^n\|_{-1,h}=\sup_{\substack{v_h\in V_h\\ v_h\neq0}}\frac{\left(\varepsilon_h^n,v_h\right)}{\|v_h\|_1}\le\frac{1}{\Delta t}\left(\int_{t_n}^{t_{n+1}}(s-t_n)^2ds\right)^{\frac12}\left(\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_{-1,h}^2ds\right)^{\frac12}$$

$$+\frac{1}{\Delta t}\left(\int_{t_n}^{t_{n+1}}ds\right)^{\frac12}\left(\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_{-1,h}^2ds\right)^{\frac12}$$
$$+\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0$$
$$+\|a_0\|_{L^\infty(\Omega)}\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0,$$

$$\|\varepsilon_h^n\|_{-1,h}^2\le\left[\frac{\sqrt{\Delta t}}{\sqrt3}\left(\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_{-1,h}^2ds\right)^{\frac12}\right.$$
$$+\frac{1}{\sqrt{\Delta t}}\left(\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_{-1,h}^2ds\right)^{\frac12}$$
$$\left.+\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0+\|a_0\|_{L^\infty(\Omega)}\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0\right]^2,$$

$$\|\varepsilon_h^n\|_{-1,h}^2\le C_2\left[\Delta t\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_{-1,h}^2ds+\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_{-1,h}^2ds\right.$$
$$\left.+\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0^2+\|a_0\|_{L^\infty(\Omega)}^2\left\|\Pi_{1,h}^k\left(u(t_{n+1})-u(t_n)\right)\right\|_0^2\right],$$

$$\le C_2\left[\Delta t\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_0^2ds+\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_0^2ds\right.$$
$$+$$
$$\left.\left(1+\|a_0\|_{L^\infty(\Omega)}^2\right)\Delta t\int_{t_n}^{t_{n+1}}\left\|\Pi_{1,h}^k\frac{\partial u}{\partial t}(s)\right\|_0^2ds\right],$$

$$\|\varepsilon_h^n\|_{-1,h}^2\le C_2\left[\Delta t\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_0^2ds+\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_0^2ds\right.$$
$$\left.\left(1+\|a_0\|_{L^\infty(\Omega)}^2\right)\Delta t\int_{t_n}^{t_{n+1}}\left\|\frac{\partial u}{\partial t}(s)\right\|_1^2ds\right],$$

since the operator $\Pi_{1,h}^k$ is uniformly bounded in $H_0^1(\Omega)$. From (7), we obtain

$$\|\eta_h^n\|_0^2\le\left[\left\|u_{0,h}-\Pi_{1,h}^k(u_0)\right\|_0^2+\frac{C_1C_2}{\varepsilon}\Delta t\sum_{n=0}^{m-1}\left\{\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_0^2ds\right.\right.$$
$$\left.\left.+C_3\Delta t\int_{t_n}^{t_{n+1}}\left(\left\|\frac{\partial u}{\partial t}(s)\right\|_1^2+\left\|\frac{\partial^2u}{\partial t^2}(s)\right\|_0^2\right)ds\right\}\right]\times\exp\left[C_1\frac{t_n}{\varepsilon}\left(1+\|a_0\|_{L^\infty(\Omega)}^2\right)\right],$$

$$\|u_h^n-u(t_n)\|_0^2\le\left\|\left(I-\Pi_{1,h}^k\right)u(t_n)\right\|_0^2+\exp(C_\varepsilon t_n)\left[\left\|u_{0,h}-\Pi_{1,h}^k(u_0)\right\|_0^2\right.$$

$$+C_\varepsilon\int_0^{t_n}\left\|\left(I-\Pi_{1,h}^k\right)\frac{\partial u}{\partial t}(s)\right\|_0^2ds$$

$$+ C_\varepsilon \left(\Delta t\right)^2 \int_0^{t_n} \left( \left\| \frac{\partial u}{\partial t}(s) \right\|_1^2 + \left\| \frac{\partial^2 u}{\partial t^2}(s) \right\|_0^2 \right) ds \Bigg],$$

where $\eta_h^n = u_h^n - \prod_{1,h}^k \left(u\left(t_n\right)\right)$.

$$\left\| u_h^n - u\left(t_n\right) \right\|_0 \le \left\| \left(I - \prod_{1,h}^k\right) u\left(t_n\right) \right\|_0 + \exp\left(C_\varepsilon t_n\right) \Big\{ \left\| u_{0,h} - \prod_{1,h}^k \left(u_0\right) \right\|_0^2$$

$$+ C_\varepsilon \int_0^{t_n} \left\| \left(I - \prod_{1,h}^k\right) \frac{\partial u}{\partial t}(s) \right\|_0^2 ds$$

$$+ C_\varepsilon \left(\Delta t\right)^2 \int_0^{t_n} \left( \left\| \frac{\partial u}{\partial t}(s) \right\|_1^2 + \left\| \frac{\partial^2 u}{\partial t^2}(s) \right\|_0^2 \right) ds \Big\}^{\frac{1}{2}}.$$

## III. CONCLUSION

This paper has presented parabolic advection-diffusion problem. Then, we study numerical schemes for initial boundary value problem. And then, we prove convergence of the semi implicit problem.

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