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The Existence of Arbitrarily Tough and Triangle-Free Graphs

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Abstract -- In this paper we mention vertex connectivity and independence number. We establish that every hamiltonian graph and any Gl graph are 1-tough. And then, we describe the bound of the toughness t(G) in terms of independence number β(G) and the number of vertices, n in G. Finally, a 1- tough graph Gl, it is shown that and the result reveals that a triangle- free graph with are obtained.

Indexed Terms - connectivity, independence number, minimum degree, layers of G, 1-tough, Hamiltonian graph, complete bipartite,triangle-free graph.

I. INTRODUCTION

A graph $G=(V(G),E(G))$ with n vertices and m edges consists of a vertex set $V(G)=\{v_1,v_2,...,v_n\}$ and an edge set $E(G)=\{e_1,e_2,...,e_m\}$ where each edge consists of two vertices called its end-vertices. If uv$\in$ E(G), then u and v are adjacent . The ends of an edge are said to be incident with the edge. The number of vertices of G is called the order of G, is denoted by υ(G). Two vertices u and v of G are said to be connected if there is a (u,v)-path in G.

A graph is said to be connected if every two of its vertices are connected; otherwise it is disconnected. A graph is simple if it has no loops and no parallel edges. The degree of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote by δ(G) and Δ(G) the minimum and maximum degrees, respectively of vertices of G. A complete graph G is a simple graph in which every pair of vertices is adjacent. If a complete graph G has n vertices, then it will be denoted by K_n . A spanning subgraph of G is a subgraph H with V(H) = V(G).

A walk in G is a finite sequence W = v_0 e_1 v_1 e_2 v_2 … $e_k v_k$, whose terms are alternately vertices and edges, such that, for $1\le i\le k,$ the ends of e_i are v_{i-1} and v_i. We say that W is a walk from v_0 to v_k or a (v_0,v_k)-walk. The vertices v_0 and v_k are called the origin and terminus of W, respectively and v_1, v_2, …, v_{k-1} its internal vertices. The integer k is the length of W. If all the edges of a walk are distinct, then it is called a trail. If, in addition, the vertices are distinct, W is called a path. Suppose that V' is a nonempty subset of V. A cycle is a closed trail in which all the vertices are distinct, except that the first vertex equals the last vertex. The component of a graph G is the maximal connected subgraph of G. We denote the number of components of G by ω(G). The ith neighborhood of v is $N_i(v)=\{u\in V\,|\,d(u,v)=i\}.$ We set $N_0(v)=\{v\}$ and abbreviate $N_1(v)$ to N(v) and call it the neighbor of v.

The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of G. Now suppose that E' is a nonempty subset of E. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E', is called the subgraph of G induced by E' and is denoted by $G[E']$; $G[E']$ is an edge - induced subgraph of G.

The vertex - connectivity or simply the connectivity κ(G) of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete , and κ(G) = n – 1 if G = K_n for some positive integer n . Hence κ(G) is the minimum number of vertices whose removal results in a disconnected or trivial graph. If G is either trivial or disconnected, κ(G) = 0. G is said to be k - connected if $\kappa(G)\ge k$. All non-trivial connected graphs are 1 - connected.

The edge-connectivity $\kappa'(G)$ of a graph G is the minimum cardinality of an edge-cut of G if G is non-trivial, and $\kappa'(K_1) = 0$. So $\kappa'(G)$ is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Thus $\kappa'(G) = 0$ if and only if G is disconnected or trivial; while $\kappa'(G) = 1$ if and only if G is connected . A graph G is k - edge - connected, $k \geq 1$, if $\kappa'(G) \geq k$.

A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph . A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y : if |X| = m and |Y| = n, such a graph is denoted by $K_{m,n}$.

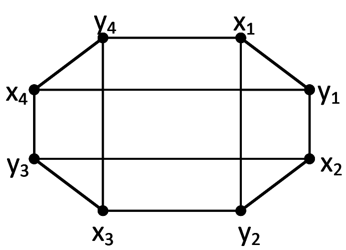

Figure 1 . (a) bipartite graph

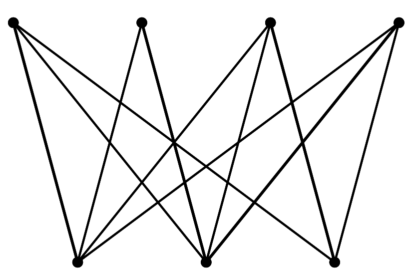

Figure 1 (b) complete bipartite graph $K_{4,3}$

A path that contains every vertex of G is called a Hamilton path of G; similarly, a Hamilton cycle of G is a cycle that contains every vertex of G. A graph is hamiltonian if it contains a Hamilton cycle.

II. THE t-TOUGH AND TRIANGLE-FREE GRAPH

A parameter that plays an important role in the study of toughness is the independence number. Two vertices that are not adjacent in a graph G are said to be independent. A set S of vertices is independent if every two vertices of S are independent. The vertex independence number or simply the independence number $\beta(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G. Let F be a graph. A graph G is F-free if G contains no induced subgraph isomorphic of F. A $K_{1,3}$-free graph is also referred to as a claw-free graph. If G is a noncomplete graph and t is a nonnegative real number such that $t \leq \frac{|S|}{\omega(G-S)}$ for every vertex-cut S of G, then G is defined to be t-tough. If G is a t-tough graph and s is a nonnegative real number such that $s < t$, then G is also s-tough. The maximum real number t for which a graph G is a t-tough is called the toughness of G and is denoted by t(G). Since complete graphs do not contain vertex-cuts, this definition does not apply to such graphs.

Consequently, we define $t(K_n) = +\infty$ for every positive integer n. Certainly, the toughness of a noncomplete graph is a rational number. Also $t(G) = 0$ if and only if G is a disconnected. It follows that if G is a noncomplete graph, then

$$t(G) = \min \frac{|S|}{\omega(G-S)},$$

where the minimum is taken over all vertex-cuts S of G.

Let G be a graph with vertices $v_1, v_2, ..., v_n$, and let $\ell \geq 1$ be an integer. We begin by defining the graph G_ℓ which is constructed by layering G ℓ times. For each k, $1 \leq k \leq \ell$, the kth layer of G_ℓ will induce a complete bipartite graph with bipartition sets $\{u_{k,1}, u_{k,2}, ..., u_{k,n}\}$ and $\{w_{k,1}, w_{k,2}, ..., w_{k,n}\}$. (See Figure 2, in which the layers of G_ℓ are schematically illustrated.) We will denote the set of vertices $\{u_{k,j} / 1 \leq k \leq \ell, 1 \leq j \leq n\}$ as Top, and the remaining set of vertices as Bottom. For $1 \leq j \leq n$, the set of vertices $\{u_{1,j}, u_{2,j}, ..., u_{\ell,j}\}$ (respectively, $\{w_{1,j}, w_{2,j}, ..., w_{\ell,j}\}$) will be called the jth top (respectively jth bottom).

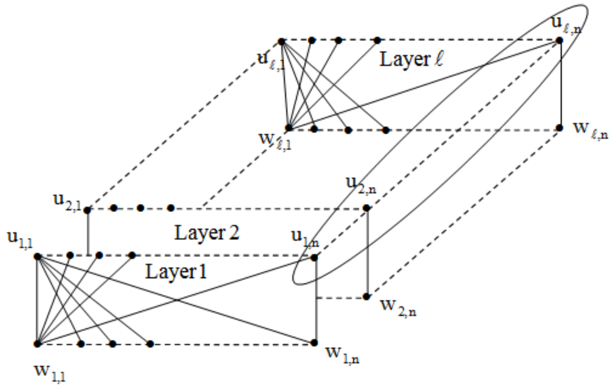

Figure 2 The Layers of G_ℓ

2. 1 Proposition

If G is a hamiltonian graph, then for any subset S of V, $\omega(G-S) \leq |S|$.

Proof:

For the spanning cycle C of G it is true that $\omega(C-S) \leq |S|$. But C being a spanning subgraph of G $\omega(G-S) \leq \omega(C-S)$. Hence the results.

2.2 Corollary

Every hamiltonian graph is 1-tough.

Proof:

Let G be a hamiltonian graph. By proposition 2.1, for any subset S of V, $\omega(G-S) \leq |S|$. It is also true for any vertex-cut S. Thus,

$$t(G) = \min \frac{|S|}{\omega(G-S)} \geq 1.$$

Hence G is 1-tough.

2.3 Lemma

Let G be a 1-tough graph on $n \geq 2$ vertices, and let $A, B \subseteq V(G)$ with $|A|+|B| \geq n+1$. Then some vertex in A is adjacent to a vertex in B.

Proof

We may assume $A \cap B$ is an independent set in G, since otherwise we are done. Let $s = |A \cap B| \geq 1$, $a = |A-B|$, $b = |B-A|$, and $c = |V(G)-(A \cup B)|$.

By assumption, $|A|+|B| = (a+s)+(b+s) \geq n+1$, and so $a+b+s \geq n-s+1$. But $a+b+c+s = n$, and thus $c \leq s-1$.

If a vertex in $A \cap B$ has a neighbor in $A \cup B$, we would be done, and thus we may assume $N(A \cap B) \subseteq V(G)-(A \cup B)$ and $c \geq 1$. Setting $X = V(G)-(A \cup B)$, we have that $\omega(G-X) \geq |A-B| = s \geq 2$, while $|X| = c \leq s-1$. This contradicts the assumption that G is 1-tough.

2.4 Lemma

Let G be a connected graph on $n \geq 2$ vertices. Suppose we obtain G_ℓ by Layering G ℓ times. Then G_ℓ is 1-tough.

Proof:

Let $X \subseteq V(G_\ell)$ such that $\omega(G_\ell - X) > 1$ and $t(G_\ell) = \dfrac{|X|}{\omega(G_\ell - X)}$. Let V_i denote the vertices in the ith layer of G_ℓ, $X_i = X \cap V_i$, and $\omega_i = \omega(\langle V_i - X_i \rangle)$. If $X_i = \varnothing$ for some i, then immediately $G_\ell - X$ is connected, contradicting $\omega(G_\ell - X) > 1$. Hence we may assume $|X_i| \geq 1$ for all i. Since $\langle V_i \rangle = K_{n,n}$ is 1-tough, we have $\omega_i \leq |X_i|$, for all i. But then

$$|X| = \sum_{i=1}^{\ell} |X_i| \geq \sum_{i=1}^{\ell} |\omega_i| \geq \omega(G_\ell - X),$$

and thus G_ℓ is 1-tough.

2.5 Theorem

Let G be a 1-tough graph on $n \geq 2$ vertices. Form G_ℓ by layering G ℓ times. Let $n_\ell = |V(G_\ell)|$ and $\beta_\ell = \beta(G_\ell)$. Then $t(G_\ell) \geq \sqrt{\frac{n_\ell}{2\beta_\ell}}$.

Proof:

Set $t = t(G_\ell)$ and assume $t < \sqrt{\frac{n_\ell}{2\beta_\ell}}$.

Let $X \subseteq V(G_\ell)$ such that $\omega(G_\ell - X) > 1$ and $t = \frac{|X|}{\omega(G_\ell - X)}$. By Lemma 2.4 we can assume $t \geq 1$. Let V_k denote the vertices in the k th layer of G_ℓ and let $X_k = X \cap V_k$. We assume that X_1 has the minimum number of vertices among the X_k.

Since $\sum_{k=1}^{\ell} |X_k| = |X| = t.\omega(G_\ell - X)$,

we have $\frac{|X_1|}{\omega(G_\ell - X)} \leq \frac{t}{\ell}$. (1)

Claim 1. $|X_1|$ satisfies $1 \leq |X_1| < \frac{n}{t} \leq n$.

Proof of Claim 1. If $|X_1| = 0$, then obviously $\omega(G_\ell - X) = 1$, a contradiction. So we have $|X_1| \geq 1$. If $|X_1| \geq \frac{n}{t}$, then by (1) we have

$$\frac{t}{\ell} \geq \frac{|X_1|}{\omega(G_\ell - X)} \geq \frac{\frac{n}{t}}{\beta(G_\ell)} = \frac{n}{t\beta_\ell}.$$

Thus $t^2 \geq \frac{\ell n}{\beta_\ell} = \frac{n_\ell}{2\beta_\ell}$, and hence $t \geq \sqrt{\frac{n_\ell}{2\beta_\ell}}$, contradicting the assumption. Since $t \geq 1$, we have $\frac{n}{t} \leq n$.

This proves Claim 1.

Since $|X_1| < n$ clearly $V_1 - X_1$ contains vertices from both Top and Bottom, and the vertices in $V_1 - X_1$ and belong to a single component of $G_\ell - X$. Henceforth, we will denote this component as H.

Let us now partition the layer numbers $\{1, 2, ..., \ell\}$ into two sets Small and Big as follows: for $1 \leq j \leq \ell, j \in \text{Small}$ (respectively, $j \in \text{Big}$) if $|X_j| \leq n-1$ (respectively, $|X_j| \geq n$). Note that $1 \in \text{Small}$ by Claim 1.

Claim 2. $\bigcup_{j \in \text{Small}} (V_j - X_j)$(i.e the vertices which remain in the small layers when X is removed) all belong to the component H.

Proof of Claim 2. If $j \in \text{Small} - \{1\}$, then certainly all vertices in $V_j - X_j$ will belong to the same component of $G_\ell - X$, since (1) $|V_j - X_j| = 2n - |X_j| \geq n+1$, and so $V_j - X_j$ contains vertices from both Top and Bottom. Thus it suffices to show there is an edge between $V_1 - X_1$ and $V_j - X_j$.

Since $V_j - X_j \geq n+1$ and $V_1 - X_1 \geq n+1$ we have

$$\begin{aligned} &|(V_j - X_j) \cap \text{Top}| + |(V_1 - X_1) \cap \text{Top}| \\ &+ |(V_j - X_j) \cap \text{Bottom}| + |(V_1 - X_1) \cap \text{Bottom}| \\ &= |V_j - X_j| + |V_1 - X_1| \geq 2n + 2, \end{aligned}$$

so either $|(V_j - X_j) \cap \text{Top}| + |(V_1 - X_1) \cap \text{Top}| \geq n+1$ or

$$|(V_j - X_j) \cap \text{Bottom}| + |(V_1 - X_1) \cap \text{Bottom}| \geq n+1.$$

Let us assume the former and define

$A=(V_j-X_j)\bigcap Top$ and $B=(V_1-X_1)\bigcap Top$. Thus we have $|A|+|B|\geq n+1.$ Since G is 1-tough, it follows by Lemma 2.3 (thinking of A,B as subsets of V(G)) that some vertex in A is adjacent to some vertex in B.

This proves Claim 2.

We now know that all vertices in $\bigcup_{j\in Small}(V_j-X_j)$ belong to the component H of $G_\ell-X.$

Let us now turn to a consideration of the layers whose indices are in Big, recalling that $j\in Big$ implies $|X_j|\geq n$. We know

$$t=\frac{|X|}{\omega(G_\ell-X)}\geq\frac{|X_1|+|U_{j\in Big}X_j|}{\omega(G_\ell-X)}\geq\frac{|X_1|+|Big|.n}{\omega(G_\ell-X)}.$$

However, if $k\in Big,$ then by Lemma 2.4 the maximum number of vertices (and hence components) in V_k-X_k that lie outside H is $|X_1|$.

Hence $\omega(G_\ell-X)\leq 1+|Big|.|X_1|,$ which gives $t\geq\frac{(|X_1|+|Big|.n)}{(1+|Big|.|X_1|)}$. Since $n>t|X_t|$ by Claim 1, we conclude

$$|Big|\leq\frac{t-|X_1|}{n-t|X_1|}. \quad (2)$$

If $\{X_1\}\geq t,$ then by (2) we have $|Big|\leq 0,$ so we have only the component H in $G_\ell-X$, contradicting $\omega(G_\ell-X)>1.$ So we can assume $|X_1|<t$.

If $n>t^2$ (equivalently $\frac{n}{t}-|X_1|>t-|X_1|$), then by (2) we find

$$|Big|\leq\frac{1}{t}\cdot\frac{t-|X_1|}{\frac{n}{t}-|X_1|}<\frac{1}{t}\leq 1,$$

implying the contradiction $|Big|=0.$ But if $n\leq t^2$, then since $\beta_\ell\leq\ell\beta(G)$ and $t<\sqrt{\frac{n_\ell}{2\beta_\ell}}$, we obtain $n\leq t^2<\frac{n_\ell}{2\beta_\ell}\leq\frac{2\ell n}{2\ell\beta(G)}=\frac{n}{\beta(G)},$ a contradiction since $\beta(G)\geq 1$.

This completes the proof of Theorem 2.5.

2.6 Lemma [5]

Let G be a graph with n vertices and independence number β. Let G_ℓ be constructed by layering G ℓ times. Then $\beta(G_\ell)\leq 2n+(\ell-2)\beta.$

2.7 Corollary

Let G be a 1-tough graph on $n\geq 2$ vertices with independence number β . Form G_ℓ by layering G $\ell\leq 2((\frac{n}{\beta})+1)$ times. Then $t(G_\ell)\geq(\frac{1}{2})\sqrt{\ell}$.

Proof:

By Lemma 2.6 we have $\beta(G_\ell)\leq 2n+(\ell-2)\beta$. Since $\ell\leq 2((\frac{n}{\beta})+1),$ this gives $\beta(G_\ell)\leq 4n.$ Now use Theorem 2.5 and $|V(G_\ell)|=n_\ell=2\ell n$ to obtain $t(G_\ell)\geq\frac{1}{2}\sqrt{\ell}$.

2.8 Lemma

Let G be a triangle-free graph. Then $\beta(G)+\kappa(G)\geq 2\delta(G).$

Proof:

This is obvious if $\kappa(G)=\delta(G)$ since $\beta(G)\geq\delta(G).$ If $\kappa(G)<\delta(G),$ let X be a vertex - cut of cardinality $\kappa(G)$ and let G_1,G_2 be two of the components of $G-X.$ Since every vertex outside X has a neighbor outside X, every component of $G-X$ has an edge. Consider the endvertices of edge vw in G_i. Since they have no common neighbor in X, one of them must have at most $(\frac{1}{2})\kappa(G)$ neighbors in X. Hence $\beta(G_i) \geq \delta(G) - (\frac{1}{2})\kappa(G),$ implying $\beta(G) \geq \beta(G_1) + \beta(G_2) \geq 2\delta(G) - \kappa(G).$

III. CONCLUSION

We conclude that a connected graph G have layering G_l,must be 1-tough.And then a 1-tough graph $G_l$have the toughness $t(G_\ell) \geq \sqrt{\frac{n_\ell}{2\beta_\ell}}$. Finally we discusse a triangle- free ghaph G ,which has the bound of minimum degree δ(G) in terms of independence number β(G) and connectivity **κ**(G).

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