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Connected Graph with Trees

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Abstract -- In this paper we mention cut vertex and cut edge in a connected graph. We establish a minimally connected graph with no cycles. And then, a graph G with n vertices, n-1 edges and no cycles, it is connected. Finally, G contains trees, whose minimum degree, δ(G) ≥ k and it is shown that the order of sub graph tree with at most δ(G)+1.

Indexed Terms: cut vertex, cut edge, vertex- cut, edge- cut, cyclic edge, components, cycle, path, tree, minimally connected

I. INTRODUCTION

A graph $G=(V(G),E(G))$ with n vertices and m edges consists of a vertex set $V(G)=\{v_1,v_2,...,v_n\}$ and an edge set $E(G)=\{e_1,e_2,...,e_m\}$ where each edge consists of two vertices called its end-vertices. We write uv for an edge $e=\{u,v\}$. If $uv\in E(G)$, then u and v are adjacent. The ends of an edge are said to be incident with the edge. The number of vertices of G is called the order of G, is denoted by $n(G)$. A graph is finite if its vertex set and edge set are finite. A graph with no edges is called an empty graph. We call a graph with just one vertex trivial and all other graphs nontrivial. A loop is an edge whose endpoints are equal. Parallel edges or multiple edges are edges that have the same pair of endpoints. A graph is simple if it has no loops and no parallel edges, A graph H is a sub graph of G if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$.

The component of a graph G is the maximal connected sub graph of G. We denote the number of components of G by $\omega(G)$. The degree $d_G(v)$ (or valency) of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively of vertices of G. A vertex of degree zero is called an isolated vertex. A vertex of degree one is called a pendant vertex.

A walk in G is a finite sequence $W=v_0e_1v_1e_2v_2...e_kv_k$, whose terms are alternately vertices and edges, such that, for $1\le i\le k$, the ends of e_i are v_{i-1} and v_i. We say that W is a walk from v_0 to v_k or a (v_0,v_k)-walk. The vertices v_0 and v_k are called the origin and terminus of W, respectively and $v_1,v_2,...,v_{k-1}$ its internal vertices. The integer k is the length of W. If all the edges of a walk are distinct, then it is called a trail. If, in addition, the vertices are distinct, W is called a path.

The length of a path is the number of edges in that path. A walk is closed if its origin and terminus are the same. A cycle is a closed trail in which all the vertices are distinct, except that the first vertex equals the last vertex.

An acyclic graph is one that contains no cycles. A tree is a connected acyclic graph.

Two vertices u and v of G are said to be connected if there is a (u,v)-path in G. A graph is said to be connected if every two of its vertices are connected; otherwise it is disconnected. The vertex-connectivity or simply the connectivity $\kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G)=n-1$ if $G=K_n$ for some positive integer n. Hence $\kappa(G)$ is the minimum number of vertices whose removal form G results in a disconnected or trivial graph. If G is either trivial or disconnected, $\kappa(G)=0$. G is said to be k-connected if $\kappa(G)\ge k$. All non-trivial connected graphs are 1-connected. An **acyclic** graph is one that contains no cycles. A **tree** is a connected acyclic graph.

II. CUT VERTEX AND CUT EDGE IN A CONNECTED GRAPH

A vertex v of a graph G is a cut vertex of G if $\omega(G - v) > \omega(G).$ An edge e of a graph G is a cut edge of G if $\omega(G - e) > \omega(G).$ A vertex -cut in a graph G is a set U of vertices of G such that $G - U$ is disconnected. A complete graph has no vertex-cut. Every graph that is not complete has a vertex-cut. Indeed, the set of all vertices distinct from two nonadjacent vertices is a vertex-cut.

An edge-cut in a graph G is a set X of edges of G such that $G - X$ is disconnected. An edge-cut X is minimum if no proper subset of X is also an edge-cut. If X is a minimum edge-cut of a connected graph G, then, necessarily $G - X$ contains exactly two components. Every non-trivial graph has an edge-cut.

2.1 Theorem

For a connected graph G, the following statements are equivalent:

(i) v is a cut vertex .

(ii) The vertex subset $V-\{v\}$ can be partitioned as $U \cup W$ such that for any $u \in U$ and any $w \in W$ every (u, w)-path passes through v.

(iii) There exist vertices u, $w \in V-\{v\}$ such that every (u, w)-path in G passes through v.

Proof

(i) $\Rightarrow$ (ii) :

Since v is a cut vertex, G – v is disconnected. Let $G_1, G_2, \ldots, G_k$ be the components of G – v.

Let $U = V(G_1)$ and $W = \bigcup_{i=2}^{k} V(G_i)$. Let $u \in U$ and $w \in W$. Specifically, let $w \in V(G_i)$ ($i \neq 1$). If there is a (u , w) - path P in G not passing through v , then P connects u and w in G – v also. Thus $G_i \cup G_i$ is a single component in G – v, contradicting our assumption. Thus every (u , w) - path in G passes through v and U and W satisfy the condition (ii).

(ii) $\Rightarrow$ (iii) : Obvious .

(iii) $\Rightarrow$ (i) :

Since every (u, w)-path in G passes through v, there is no (u, w)-path in G–v. Thus u and w belong to different components of G–v. That is G–v is disconnected and v is a cut vertex of G.

2.2 Theorem

For a connected graph G, the following statements are equivalent:

(i) e is a cut edge of G .

(ii)If e = ab, there is a partition of the edge subset E – {e} as $E_1 \cup E_2$ with a $\in V([E_1])$ and b $\in V([E_2])$ such that for any $u \in V([E_1])$ and any w $\in V([E_2])$ every (u , w)-path contains e.

(iii) There exist vertices u and w such that every (u, w) - path in G contains e.

(iv) e is not a cyclic edge of G .

Proof

(i) $\Rightarrow$ (ii) :

Let G_1 and G_2 be the two components of G−e and $E_1 = E(G_1)$ and $E_2 = E(G_2)$. If $u \in V(G_1)$ and $w \in V(G_2)$ exist such that there is a (u, w)-path P in G which does not contain e, then u and w are connected in G–e by the path P. This means that $G_1 \cup G_2$, that is G–e, is connected, contradicting the hypothesis.

(ii) $\Rightarrow$ (iii) : Obvious.

(iii) $\Rightarrow$ (iv) :

We prove the contra - positive. Suppose e lies on a cycle C. Then C–e gives an (a, b)-path Q not containing e. With vertices u and w following the condition given in statement (iii), let P be any (u, w)-path. Without loss of generality let us assume that a and b occur in that order in P. Let u_0 and w_0 be the first and last vertices that P has in common with C (the possibility of these coinciding with a , b , u or w is not ruled out).

Then $P_{u,u_0} \cup Q_{u_0,w_0} \cup P_{w_0,w}$ is a (u, w) -path P' of G which does not contain e, contradicting (iii). (See Figure 1)

(iv) $\Rightarrow$(i) :

To prove the contra - positive suppose G–e is connected. Then there is an (a ,b)- path P in G–e. But then $P \cup e$ is a cycle containing e. This contradicts (iv).

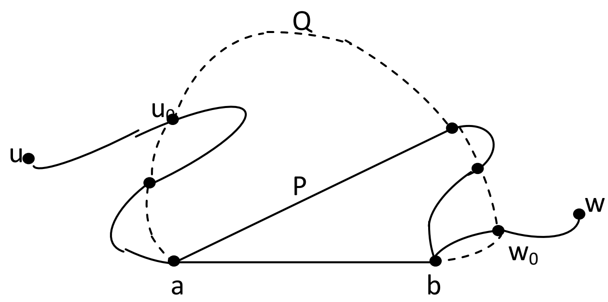

Figure 1.

III. MINIMALLY CONNECTED WITH A TREE

A graph is said to be minimally connected if removal of any one edge from it disconnects the graph. Clearly, a minimally connected graph has no cycles.

3.1 Theorem
A graph is a tree if and only if there is exactly one path between every pair of its vertices.

Proof
Let G be a graph and let there be exactly one path between every pair of vertices in G. So G is connected. Now G has no cycles, because if G contains a cycle, say between vertices u and v, then there are two distinct paths between u and v, which is a contradiction. Thus G is connected and is without cycles, therefore it is a tree.

Conversely, let G be a tree. Since G is connected, there is at least one path between every pair of vertices in G. Let there be two distinct paths between two vertices u and v of G. The union of these two paths contains a cycle which contradicts the fact that G is tree. Hence there is exactly one path between every pair of vertices of a tree.

3.2 Theorem
A tree with n vertices has n–1 edges.

Proof
We prove the result by using induction on n, the number of vertices. The result is obviously true for n=1,2 and 3. Let the result be true for all trees with fewer than n vertices. Let T be a tree with n vertices and let e be an edge with end vertices u and v. So the only path between u and v is e. Therefore deletion of e from T disconnects T. Now, T– e consists of exactly two components T_1 and T_2 say, and as there were no cycles to begin with, each component is a tree. Let n_1 and n_2 be the number of vertices in T_1 and T_2 respectively, so that $n_1 + n_2 = n$. Also, $n_1 < n$ and $n_2 < n$. Thus, by induction hypothesis, number of edges in T_1 and T_2 are respectively $n_1 - 1$ and $n_2 - 1$.

Hence the number of edges in

$$T = n_1 - 1 + n_2 - 1 + 1$$
$$= n_1 + n_2 - 1$$
$$= n - 1.$$

3.3 Theorem
Any connected graph with n vertices and n–1 edges is a tree.

Proof
Let G be a connected graph with n vertices and n–1 edges. We show that G contains no cycles. Assume to the contrary that G contains cycles.

Remove an edge from a cycle so that the resulting graph is again connected. Continue this process of removing one edge from one cycle at a time till the resulting graph H is a tree. As H has n vertices, so number of edges in H is n–1. Now the number of edges in G is greater than the number of edges in H. So $n - 1 > n - 1$, which is not possible. Hence, G has no cycles and therefore is a tree.

3.4 Theorem
A graph is a tree if and only if it is minimally connected.

Proof

Let the graph G be minimally connected. Then G has no cycles and therefore is a tree.

Conversely, let G be a tree. Then G contains no cycles and deletion of any edge from G disconnects the graph. Hence G is minimally connected.

The following results give some more properties of trees.

3.5 Theorem

A graph G with n vertices, n–1 edges and no cycles is connected.

Proof

Let G be a graph without cycles with n vertices and n–1 edges. We have to prove that G is connected. Assume that G is disconnected. So G consists of two or more components and each component is also without cycles. We assume without loss of generality that G has two components, say G_1 and G_2 . Add an edge e between a vertex u in G_1 and a vertex v in G_2. Since there is no path between u and v in G, adding e did not create a cycle. Thus $G \bigcup e$ is a connected graph (tree) of n vertices, having n edges and no cycles. This contradicts the fact that a tree with n vertices hasn–1 edges. Hence Gis connected.

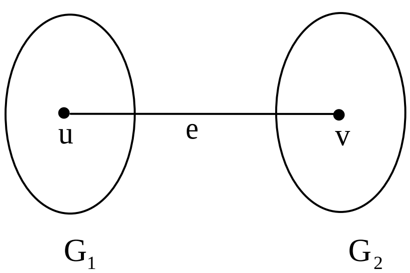

Figure. 2

3.6 Theorem

Any tree with at least two vertices has at least two pendant vertices.

Proof

Let the number of vertices in a given tree T be $n(n>1)$. So the number of edges in T is n–1. Therefore the degree sum of the tree is $2(n-1)$. This degree sum is to be divided among the n vertices. Since a tree is connected it cannot have a vertex of 0 degree. Each vertex contributes at least 1 to the above sum. Thus there must be at least two vertices of degree exactly 1.

Alternative Proof 1

We use induction on n. The result is obviously true for all trees having fewer than n vertices. We know that T has n–1 edges, and is every edge of T is incident with a pendant vertex, then T has at least two pendant vertices, and the proof is complete. So let there be some edge of T that is not incident with a pendant vertex and let this edge be e = uv (Figure.3). Removing the edge e, we see that the graph T – e consists of a pair of trees say T_1 and T_2 with each having fewer than n-vertices. Let $u \in V(T_1), v \in V(T_2)$, and $|V(T_1)| = n_1, |V(T_2)| = n_2$. Applying induction hypothesis on both T_1 and T_2 , we observe that each of T_1 and T_2 has two pendant vertices. This shows that each of T_1 and T_2 has at least one pendant vertex that is not incident with the edge e. Thus the graph $T-e+e=T$ has at least two pendant vertices.

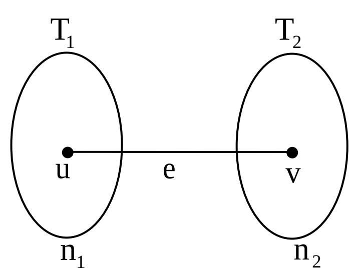

Figure 3

Alternative Proof 2

Let T be a tree with $n(n>1)$ vertices. The number of edges in T is n–1 and the sum of degrees in T is $2(n-1)$, that is , $\sum d_i = 2(n-1)$. Assume T has exactly one vertex v_1 of degree one, while all the other

n –1 vertices have degree ≥ 2. Then sum of degrees is

$$d(v_1)+d(v_2)+\cdots+d(v_n) \geq 1+2+2+\cdots+2 = 1+2(n-1).$$

So, $2(n-1) \ge 1+2(n-1)$, implying $0 \ge 1$, which is absurd. Hence at least two vertices of degree one.

3.7 Theorem

The sequence $[d_i]_1^n$ of positive integers is a degree sequence of a tree if and only if

(i) $d_i > 1$ for all $i, 1 < i < n$ and

(ii) $\sum_{i=1}^{n} d_i = 2n-2.$

Proof

Since a tree has no isolated vertex, therefore $d_i \ge 1$ for i. Also ,

$\sum_{i=1}^{n} d_i = 2(n-1)$, as a tree with n vertices has n–1 edges.

We use induction on n. For n = 2, the sequence is [1,1] and is obviously the degree sequence of K_2. Suppose the claim is true for all positive sequences of length less than n.

Let $\sum_{i=1}^{n} d_i$ be the non-decreasing positive sequence of n terms, satisfying conditions (i) and (ii).

Then $d_1 = 1$ and $d_n > 1$.

Now, consider the sequence $D' = [d_2, d_3, \cdots, d_{n-1}, d_n - 1]$, which is a sequence of length n–1. Obviously in D', $d_i \ge 1$ and

$$\sum_i d_i = d_2 + d_3 + \cdots + d_{n-1} + d_n - 1$$
$$= d_1 + d_2 + d_3 + \cdots + d_{n-1} + d_n - 1 - 1$$
$$= 2n - 2 - 2$$
$$= 2(n-1) - 2.$$

So D' satisfies conditions (i) and (ii), and by induction hypothesis there is a tree T' realizing D'. In T', add a new vertex and join it to the vertex having degree d_n–1 to get a tree T. Therefore the degree sequence of T is $[d_1, d_1, \cdots, d_n]$.

3.8 Theorem

A forest of k trees which have a total of n vertices has n–k edges.

Proof

Let G be a forest and $T_1, T_2, \ldots, T_k$ be the k trees of G. Let G have n vertices and $T_1, T_2, \ldots, T_k$ have respectively $n_1, n_2, \ldots, n_k$ vertices.

Then $n_1 + n_2 + \ldots + n_k = n$. Also, the number of edges in $T_1, T_2, \ldots, T_k$ are respectively $n_1 - 1, n_2 - 1, \ldots, n_k - 1$.

Thus number of edges in $G = n_1 - 1 + n_2 - 1 + \ldots + n_k - 1 = n_1 + n_2 + \ldots + n_k - k = n - k.$

3.9 Theorem

Let T be a tree with k edges. If G is a graph whose minimum degree satisfies $\delta(G) \ge k$, then G contains T as a subgraph. Alternatively, G contains every tree of order at most $\delta(G)+1$ as a subgraph.

Proof

We use induction on k. If k = 0, then T = K_1 and it is clear that K_1 is a subgraph of any graph. Further, if k = 1, then T = K_2 and K_2 is a subgraph of any graph whose minimum degree is one. Assume the result is true for all trees with k – 1 edges $(k \ge 2)$ and consider a tree T with exactly k edges. We know that T contains at least two pendant vertices. Let v be one of them and let w be the vertex that is adjacent to v. Consider the graph T – v , Since T – v has k – 1 edges, the induction hypothesis applies, so T – v is a subgraph of G. We can think of

T – v as actually sitting inside G (meaning w is a vertex of G, too). Since G contains at least k + 1 vertices, and

T – v contains k vertices, there exist vertices of G that are not a part of the subgraph T – v. Further, since the degree of w in G is at least k, there must be a vertex u not in T – v that is adjacent to w. The subgraph T – v together with u forms the tree T as a subgraph of G.

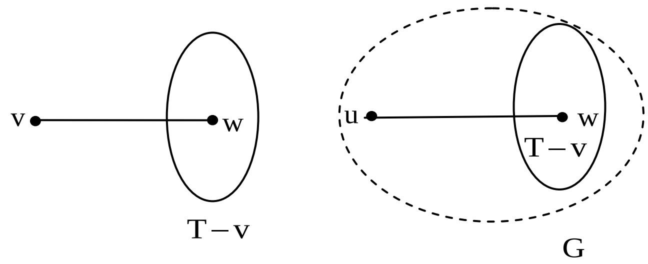

Figure.4.

IV. TREES AND COTREES OF A CONNECTED GRAPH

A tree in which one vertex (called the root) is distinguished from all the others is called a rooted tree.

A binary tree is defined as a tree in which there is exactly one vertex of degree two and each of the remaining vertices is of degree one or three. Obviously, a binary tree has three or more vertices. Since the vertex of degree two is distinct from all other vertices, it serves as a root, and so every binary tree is a rooted tree.

The height of a rooted tree is the length of a longest path from the root. If T is a binary tree of height h, then its left and right subtrees both have height less than or equal to h – 1. If H is a subgraph of G, the complement of H in G, denoted by $\bar{H}(G)$, is the subgraph G – E(H). If G is connected, a subgraph of the form $\bar{T}$, where T is a spanning tree, is called a cotree of G.

For subsets S and S' of V, we denote by $[S, S']$ the set of edges with one end in S and the other in S'. An edge cut of G is a subset of E of the form $[S, \bar{S}]$, where S is a nonempty proper subset of v and $\bar{S} = V/S$. A minimal edge cut of G is called a bond; each cut edge e, for instance, gives rise to a bond $\{e\}$. If G is connected, then a bond B of G is a minimal subset of E such that G – B is disconnected.

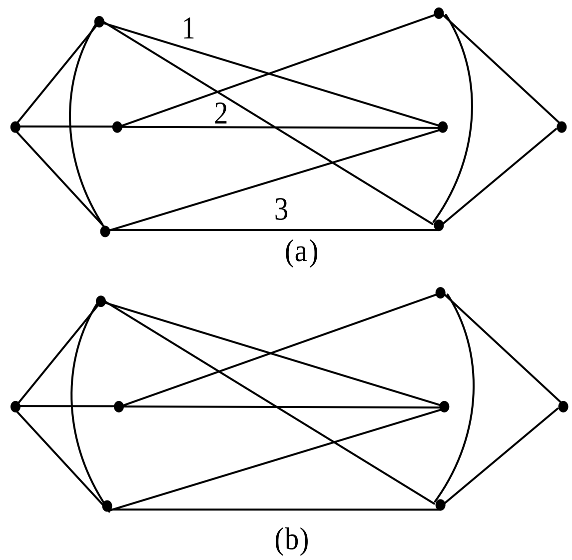

Figure 5 (a) An edge cut; (b) A bond

4.1Theorem

The binary tree of fixed height h has $2^{h+1} - 1$ vertices.

Proof

The assertion is trivially true if $h = 0$. Assume for some $k \geq 0$, that a binary tree of fixed height k has $2^{h+1} - 1$ vertices, and let T be a binary tree of height k + 1. Since T is a binary tree, its left and right subtrees, say T_1 and T_2 must be binary tree. Furthermore, trees T_1 and T_2 are both of height k, so by the induction hypothesis, they each contain $2^{h+1} - 1$ vertices. Thus, the number of vertices of T is $1 + 2^{k+1} - 1 + 2^{k+1} - 1 = 2^{k+2} - 1$.

4.2 Theorem

Every connected graph has at least one spanning tree.

Proof

Let G be a connected graph. If G has no cycles, then it is its own spanning tree. If G has cycles, then on deleting one edge from each of the cycles, the graph remains connected and cycle free containing all the vertices of G.

4.3 Theorem
Let T be a spanning tree of a connected graph G and let e be an edge of G not in T. Then T + e contains a unique cycle.

Proof
Since T is acyclic, each cycle of T + e contains e. Moreover, C is a cycle of T + e if and only if C – e is a path in T connecting the ends of e. By theorem 2.2, T has a unique such path; therefore T + e contains a unique cycle.

4.4 Theorem
Let T be a spanning tree of a connected graph G, and let e be any edge of T. Then
(i) the cotree $\overline{T}$ contains no bond of G;
(ii) $\overline{T}+e$ contains a unique bond of G.

Proof
(i)Let B be a bond of G. Then $G-B$ is disconnected, and so cannot contain the spanning tree T. Therefore B is not contained in $\overline{T}$.

(ii) Denote by S the vertex set of one of the two components of $T-e$. The edge cut $B=\left[S,\overline{S}\right]$ is clearly a bond of G, and is contained in $\overline{T}+e$. Now, for any $b\in B,\ T-e+b$ is a spanning tree of G. Therefore every bond of G contained in $\overline{T}+e$ must include every such element b. It follows that B is the only bond of G contained in $\overline{T}+e$.

II. CONCLUSION

We conclude that every connected graph has at least one spanning tree. And then, one edge added to a tree, which contains a unique cycle. Finally, the result reveals that spanning tree T of a connected graph G, its cotree $\overline{T}$ with no bound of G and $\overline{T}+e$ with a unique bound of G are obtained.

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