Von Neumann Stability Analysis for Time- Dependent Diffusion Equation

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Abstract -- In this paper, we describe two different finite difference schemes for solving the time fractional diffusion equation. And, we study the method of lines discretizations. Then, we use to check the stability of finite difference schemes by using Von Neumann analysis.

Indexed Terms: Diffusion equation, explicit method, Crank-Nicolson method, Von-Neumann analysis

I. INTRODUCTION

We study finite difference methods for time-dependent partial differential equations, where variations in space are related to variations in time.

The heat equation (or diffusion equation),

$$u_t = k\,u_{xx}. \qquad (1)$$

This is the classical example of a parabolic equation, and many of the general properties seen here carry over to the design of numerical methods for other parabolic equations. We will assume $k=1$ for simplicity but some comments will be made about how the results scale to other values of $k>0$.

Along with this equation we need initial conditions at some time t_0, which we typically take to be $t_0 = 0$,

$$u(x,0) = \eta(x) \qquad (2)$$

and also boundary conditions if we are working on a bounded domain,

for example, the Dirichlet conditions

$$u(0,t) = g_0(t) \text{ for } t>0$$
$$u(1,t) = g_1(t) \text{ for } t>0 \qquad (3)$$
$$\text{if } 0 \le x \le 1.$$

We have already studied the steady state version of this equation and spatial discretizations of u_{xx}. We have also studied discretizations of the time derivatives.

In practice we generally apply a set of finite difference equations on a discrete grid with grid points (x_i, t_n) where $x_i = ih$, $t_n = nk$.
Here $h = \Delta x$ is the mesh spacing on the *x*-axis and $k = \Delta t$ is the time step. Let $U_i^n \approx u(x_i, t_n)$ represent the numerical approximation at grid point (x_i, t_n).

Since the heat equation is an evolution equation that can be solved forward in time, we set up our difference equations in a form where we can march forward in time, determining the value U_i^{n+1} for all *i* from the values U_i^n at the previous time level, or perhaps using also values at earlier time levels with a multistep formula.

As an example, one natural discretizations of Equation (1) would be

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n). \qquad (4)$$

This uses our standard centered difference in space and a forward difference in time. This is an explicit method since we can compute each U_i^{n+1} explicity in terms of the previous data

$$U_i^{n+1} = U_i^n + \frac{k}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n). \qquad (5)$$

This is a one-step method in time, which is also called a two-level method in the context of partial differential equations since it involves the solution at two different time levels.

Another one-step method, which is much more useful in practice as we will see below, is the Crank-Nicolson method,

$$\frac{U_i^{n+1}-U_i^n}{k}=\frac{1}{2}(D^2U_i^n+D^2U_i^{n+1}) \quad (6)$$

$$=\frac{1}{2h^2}(U_{i-1}^n-2U_i^n+U_{i+1}^n+U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1})$$

which can be rewritten as for i =1,….,m.

$$U_i^{n+1}=U_i^n+\frac{k}{2h^2}(U_{i-1}^n-2U_i^n+U_{i+1}^n+U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1}) \quad (7)$$

or

$$-rU_{i-1}^{n+1}+(1+2r)U_i^{n+1}-rU_{i+1}^{n+1}=rU_{i-1}^n+(1-2r)U_i^n+rU_{i+1}^n, \quad (8)$$

For i =1,….,m.

where $r=\frac{k}{2h^2}$. This is an implicit method and gives a tridiagonal system of equations to solve for all the values U_i^{n+1} simultaneously. The boundary conditions $u(0,t)=g_0(t)$, $u(1,t)=g_1(t)$ come into these equations.

Since a tridiagonal system of m equations can be solved with $O(m)$ work, this method is essentially as efficient per time step as an explicit method. The heat equation is "stiff", and hence this implicit method, which allows much larger time steps to be taken than an explicit method, is a very efficient method for the heat equation.

II. METHOD OF LINES DISCRETIZATIONS

The time-dependent partial differential equations relates to the stability theory we have already developed for time-dependent ordinary differential equations, it is easiest to first consider the so-called Method of lines discretization of the partial differential equation.

For example, we will discredited the heat Equation (1) in space at grid point x_i by

$$U_i'(t)=\frac{1}{h^2}(U_{i-1}(t)-2U_i(t)+U_{i+1}(t)), \quad (9)$$

for $i=1,2,\cdots,m$.

where prime now means differentiation with respect to time.

If $i=1$, then $U_1'(t)=\frac{1}{h^2}(U_0(t)-2U_1(t)+U_2(t))$

$$=\frac{1}{h^2}\left(g_0(t)-2U_1(t)+U_2(t)\right).$$

If $i=2$, then $U_2'(t)=\frac{1}{h^2}\left(U_1(t)-2U_2(t)+U_3(t)\right)$.

If $i=2$, then $U_3'(t)=\frac{1}{h^2}(U_2(t)-2U_3(t)+U_4(t))$.

⋮

If $i=m$, then $U_m'(t)=\frac{1}{h^2}(U_{m-1}(t)-2U_m(t)+U_{m+1}(t))$

$$=\frac{1}{h^2}(U_{m-1}(t)-2U_m(t)+g_1(t)).$$

We can find this as a coupled system of m ordinary differential equations for the variables $U_i(t)$, which vary continuously in time along the lines . This system can be written as, $U'(t)=AU(t)+g(t)$. (10) where the tridiagonal matrix A and $g(t)$ includes the terms needed for the boundary conditions,

$$A=\frac{1}{h^2}\begin{bmatrix}-2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2\end{bmatrix}$$

$U_0(t)\equiv g_0(t)$ and $U_{m+1}(t)\equiv g_1(t)$,

$$g(t)=\frac{1}{h^2}\begin{bmatrix} g_0(t)\\ 0\\ 0\\ \vdots\\ 0\\ g_1(t)\end{bmatrix}. \qquad (11)$$

III. DEFINITION

Suppose a finite difference method for a linear boundary value problem gives a sequence of matrix equations of the form $A^h U^h = g^h$ where *h* is the mesh width. We say that the method is stable if $(A^h)^{-1}$ exists for all *h* sufficiently small (for $h < h_0$, say) and if there is a constant *C*, independent of *h*, such that

$$\|(A^h)^{-1}\| \le C \text{ fo all } h < h_0. \qquad (12)$$

IV. STABILITY IN THE 2 –Norm

If the matrix *A* from Equation (11) is symmetric, the 2-norm of *A* is equal to its spectral radius is $\| A \|_2 = \rho(A) = \max_{1\le p\le m} |\lambda_p|$, λ_p is the p^{th} eigen value of the matrix. The matrix A^{-1} is also symmetric and the eigen values of A^{-1} are simply the inverses of the eigen values of *A*.

So, $\| A^{-1} \|_2 = \rho(A^{-1}) = \max_{1\le p\le m} |(\lambda_p)^{-1}|$

$$=\left(\min_{1\le p\le m} |\lambda_p|\right)^{-1}.$$

So all we need to do is compute the eigen values of *A* and show that they are bounded away from zero as $h \to 0$.

Then the *m* eigen values of *A* are given by $\lambda_p = \frac{2}{h^2}(\cos(p\pi h)-1)$, for $p = 1, 2, \cdots, m$. (13)

The eigenvector u^p corresponding to λ_p has components u_j^p for $j = 1, 2, \cdots, m$ given by

$$u_j^p = \sin(p\pi jh) \qquad (14)$$

We can be verified by checking that $Au^p = \lambda_p u^p$.

The j^{th} component of the vector Au^p is

$$(Au^p)_j = \frac{1}{h^2}(u_{j-1}^p - 2u_j^p + u_{j+1}^p)$$

$$= \frac{1}{h^2}\left(\sin(p\pi(j-1)h) - 2\sin(p\pi jh) + \sin(p\pi(j+1)h)\right)$$

$$= \frac{1}{h^2}\left(\sin(p\pi jh)\cos(p\pi h) - \cos(p\pi jh)\sin(p\pi h)\right.$$

$$\left.-2\sin(p\pi jh) + \sin(p\pi jh)\cos(p\pi h) + \cos(p\pi jh)\sin(p\pi h)\right)$$

$$= \frac{1}{h^2}\left(\sin(p\pi jh)\cos(p\pi h) - 2\sin(p\pi jh) + \sin(p\pi jh)\cos(p\pi h)\right)$$

$$= \lambda_p u_j^p.$$

For $j = 1$ and $j = m$, the j^{th} component of Au^p looks slightly different (the u_{j-1}^p or u_{j+1}^p term is missing) but that the above form and trigonometric manipulations are still valid provided that we define $u_0^p = u_{m+1}^p = 0$,

as is consistent with Equation (14). From Equation (13) we see that the smallest eigenvalue of *A* (in magnitude) is

$$\lambda_1 = \frac{2}{h^2}(\cos(\pi h)-1)$$

$$= \frac{2}{h^2}\left(-\frac{1}{2}\pi^2 h^2 + \frac{1}{24}\pi^4 h^4 + O(h^6)\right) = -\pi^2 + O(h^2).$$

This is clearly bounded away from zero as $h \to 0$, and so we see that the method is stable in the 2-norm.

V. VON NEUMANN ANALYSIS

The Von Neumann approach to stability analysis is based on Fourier analysis and hence is generally limited to constant coefficient linear partial differential equations. For simplicity it is usually applied to the Cauchy problem, which is the partial differential equation on all space with no boundaries,

$-\infty < x < \infty$ in the one-dimensional case. Von Neumann analysis can also be used to study the stability of problems with periodic boundary conditions.

The Cauchy problem for linear partial differential equations can be solved using Fourier transforms. The basic reason this works is that the functions $e^{i\xi x}$ with wave number $\xi = \text{constant}$ are eigen functions of the differential operator ∂x, $\partial x\, e^{i\xi x} = i\xi\, e^{i\xi x}$, and hence of any constant coefficient linear differential operator. Von Neumann analysis is based on the fact that the related grid function $W_j = e^{ijh\xi}$ is an eigen function of any standard finite difference operator. For example, if we approximate $v'(x_j)$ by $D_0 V_j = \frac{1}{2h}(V_{j+1} - V_{j-1})$, then in general the grid function $D_0 V$ is not just a scalar multiple of *V*.

But for the special case of *W*, we obtain

$$D_o W_j = \frac{1}{2h}(e^{i(j+1)h\xi} - e^{i(j-1)h\xi})$$
$$= \frac{1}{2h}(e^{ih\xi} - e^{-ih\xi})e^{ijh\xi} = \frac{i}{h}\sin(h\xi)e^{ijh\xi}$$
$$= \frac{i}{h}\sin(h\xi)W_j.$$

So *W* is an "eigen gridfunciton" of the operator D_0, with eigen value $\frac{i}{h}\sin(h\xi)$.

Note the relation between these and the eigen functions and eigen values of the operator ∂x found earlier: W_j is simply the eigen function $\omega(x)$ of ∂_x evaluated at the point x_j, and for small $h\xi$ we can approximate the eigenvalue of D_0 by

$$\frac{i}{h}\sin(h\xi) = \frac{i}{h}\left(h\xi - \frac{1}{6}h^3\xi^3 + \mathrm{O}(h^5\xi^5)\right)$$
$$= i\xi - \frac{i}{6}h^2\xi^3 + \cdots.$$

This corresponds with the eigenvalue $i\xi$ of ∂_x to $\mathrm{O}(h^2\xi^3)$.

We will consider the Equation (5). To apply Von Neumann analysis we consider how this method works on a single wave number ξ, we set

$$U_j^n = e^{ijh\xi}. \qquad (15)$$

Then we expect that,

$$U_j^{n+1} = g(\xi)\, e^{ijh\xi}, \qquad (16)$$

where $g(\xi)$ is the amplification factor for this wave number.

Inserting these expressions into Equation (5) give

$$g(\xi)\, e^{ijh\xi} = e^{ijh\xi} + \frac{k}{h^2}(e^{i\xi(j-1)h} - 2e^{ijh\xi} + e^{i\xi(j+1)h})$$
$$= \left(1 + \frac{k}{h^2}(e^{-i\xi h} - 2 + e^{i\xi h})\right)e^{ijh\xi}$$

$$g(\xi) = 1 + \frac{k}{h^2}(e^{-i\xi h} - 2 + e^{i\xi h})$$
$$= 1 + \frac{k}{h^2}(-2 + 2\cos(\xi h))$$

,and hence

$$g(\xi) = 1 + \frac{2k}{h^2}(\cos(\xi h) - 1).$$

Since $-1 \le \cos(\xi h) \le 1$ for any value of ξ, we see that $1 - 4\frac{k}{h^2} \le g(\xi) \le 1$, for all ξ.

Then, $|\, g(\xi)\,| \le 1$ is equivalent to

$$|\, g(\xi)\,| = \left|1 + \frac{2k}{h^2}(\cos(\xi h) - 1)\right| \le 1$$
$$\left|1 + \frac{2k}{h^2}(-2\sin^2(\frac{\xi h}{2}))\right| \le 1,$$
$$\left|1 - \frac{4k}{h^2}\sin^2(\frac{\xi h}{2})\right| \le 1.$$

So, $-1 \le 1 - \frac{4k}{h^2}\sin^2(\frac{\xi h}{2}) \le 1$,

$$-2 \le -\frac{4k}{h^2}\sin^2\left(\frac{\xi h}{2}\right) \le 0,$$

Thus, $4\frac{k}{h^2} \le 2$, $\frac{k}{h^2} \le \frac{1}{2}$.

This is exactly the stability restriction $r = \frac{k}{h^2} \le \frac{1}{2}$.

Now we will prove that the Crank-Nicolson method is stable for all *k* and *h* can also be shown using Von Neumann analysis. Substituting Equation (15) and Equation (16) into the difference Equation (7) and cancelling the common factor $e^{ijh\xi}$ gives the following relation,

$$g(\xi)e^{ijh\xi} = e^{ijh\xi} + \frac{k}{2h^2}\left(e^{i\xi(j-1)h} - 2e^{ijh\xi} + e^{i\xi(j+1)h} + g(\xi)e^{i\xi(j-1)h} - 2g(\xi)e^{ijh\xi} + g(\xi)e^{i\xi(j+1)h}\right)$$

Then, $g = 1 + \frac{k}{2h^2}(e^{-i\xi h} - 2 + e^{i\xi h})(1+g)$,

and hence $g = 1 + \frac{z}{2}(1+g)$, $g - \frac{zg}{2} = 1 + \frac{z}{2}$.

Thus, $g(1-\frac{z}{2}) = 1 + \frac{1}{2}z$,

So, $g = \frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$,

where $z = \frac{k}{h^2}(e^{-i\xi h} - 2 + e^{i\xi h})$

$= \frac{2k}{h^2}(\cos(\xi h) - 1)$ .Since $z \le 0$ for all ξ Therefore, $|g| \le 1$ and the method is stable for any *k* and *h*.

VI. CONCLUSION

This paper has presented two finite difference methods for solving diffusion equation. Next, we proved that finite difference Crank- Nicolson method is stable by using Von Neumann analysis

REFERENCES

[1] Evans, L.C., "Partial Differential Equations", American Mathematical Society, Providence, New York, 1998.

[2] Griffits, D.F., "An Introduction to Matlab Version 2.2", University of Dundee, Ireland, 1996.

[3] Jungel, A., "Numerical Methods for Partial Differential Equations I": Elliptic and Parabolic Equations, (Lecture Notes), 2001.

[4] Mathews, J.H., "Numerical Methods for Mathematics", Science and Engineering, Prentice-Hall International Education, 1992..

[5] Smith, G.D., "Numerical Solution of Partial Differential Equations, Finite Difference Methods", 3rd Ed., Clarendon Press, Oxford, 1985.

[6] Thomas, J.W., "Numerical Partial Differential Equations, Conservation Laws and Elliptic Equations", Springer-Verlag, New York, 1998.