Signed and Roman Edge Dominating Functions of Corona Product Graph of a Cycle with a Complete Graph

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Abstract -- Graph Theory has been realized as one of the most useful branches of Mathematics of recent origin with wide applications to combinatorial problems and classical algebraic problems. Graph theory has applications in diverse areas such as social sciences, linguistics, physical sciences, communication engineering etc. The theory of domination in graphs is an emerging area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al [13, 14]. Frucht and Harary [11] introduced a new product on two graphs G1 and G2, called corona product denoted by G18G2. The object is to construct a new and simple operation on two graphs G1 and G2 called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G1 and of G2 . In this paper, some results on minimal signed and Roman edge dominating functions of corona product graph of a cycle with a complete graph are presented.

Indexed Terms: Corona Product, Cycle, Complete graph, signed edge dominating function, Roman edge dominating function

I. INTRODUCTION

Domination Theory has a wide range of applications to many fields like Engineering, Communication Networks, Social sciences, linguistics, physical sciences and many others. Allan, R.B. and Laskar, R.[1], Cockayne, E.J. and Hedetniemi, S.T. [7] have studied various domination parameters of graphs.

Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. Every branch of mathematics employs some notion of product that enables the combination or decomposition of its elemental structures.

The concept of edge domination was introduced by Mitchell and Hedetniemi [18] and it is explored by many researchers. Arumugam and Velammal [6] have discussed the edge domination in graphs while the fractional edge domination in graphs is discussed in Arumugam and Jerry [5]. The complementary edge domination in graphs is studied by Kulli and Soner [17] while Jayaram [16] has studied the line dominating sets and obtained bounds for the line domination number. The bipartite graphs with equal edge domination number and maximum matching cardinality are characterized by Dutton and Klostermeyer [10] while Yannakakis and Gavril [20] have shown that edge dominating set problem is NP-complete even when restricted to planar or bipartite graphs of maximum degree.

CORONA PRODUCT OF C_n AND K_m

The corona product of a cycle C_n with a complete graph K_m is a graph obtained by taking one copy of a n – vertex graph C_n and n copies of K_m and then joining the i^{th} vertex of C_n to every vertex of i^{th} copy of K_m . This graph is denoted by $C_n \odot K_m$.

The vertices of C_n are denoted by $v_1, v_2, \dots, v_n$. The edges in C_n are denoted by $e_1, e_2, \dots, e_n$ where e_i is the edge joining the vertices v_i and v_{i+1}, $i \neq n$. For $i = n$, e_n is the edge joining the vertices v_n and v_1.

The vertices in the i^{th} copy of K_m are denoted by $w_{i1}, w_{i2}, \dots, w_{im}$. The edges in the i^{th} copy of K_m are denoted by $l_{ij}, j = 1,2, \dots, \frac{m(m-1)}{2}$.

There are another type of edges in G denoted by $h_{ij}, i = 1,2, \dots, n$ and $j = 1,2, \dots, m$ is the edge joining the vertex v_i of C_n to vertex w_{ij} of i^{th} copy of K_m. These edges which are in G and related to the i^{th} copy of K_m are denoted by $h_{i1}, h_{i2}, \dots, h_{im}$ and these are adjacent to each other and incident with the vertex v_i of C_n.

Some properties of corona product graph $G = C_n \odot K_m$ are studied by Anita [2] and some results on minimal edge dominating sets and functions of this graph are presented in [3]. Further minimal total edge dominating sets and functions of this graph are also studied by the authors [4].

II. SIGNED EDGE DOMINATING FUNCTION

The concept of Signed dominating function was introduced by Dunbar et al., [9]. There is a variety of possible applications for this variation of domination. By assigning the values -1 or $+1$ to the vertices of a graph we can model such things as networks of positive and negative electrical charges, networks of positive and negative spins of electrons and networks of people or organizations in which global decisions can be made.

In this section, we prove some theorems on minimal signed edge dominating functions of the graph $G = C_n \odot K_m$. Let us recall the definitions of signed edge dominating function and minimal signed edge dominating function of a graph $G(V, E)$.

Definition: Let $G(V, E)$ be a graph. A function $f : E \to \{-1, 1\}$ is called a signed edge dominating function (SEDF) of G if

$$f(\mathrm{N}[e]) = \sum_{e' \in N[e]} f(e') \geq 1, \forall\, e \in E(G).$$

A signed edge dominating function f of G is called a minimal signed edge dominating function (MSEDF) if for all $g < f$, g is not a signed edge dominating function.

We need the following Theorem which is presented in [2].

Theorem 2.1: The adjacency of an edge e in $G = C_n \odot K_m$ is given by

$$adj(e) = \begin{cases} 2m+2, \text{if } e = e_i \in C_n, \\ 2m-2, \text{if } e = l_{ij} \in i^{th} \text{ copy of } K_m, \\ 2m, \text{if } e = h_{ij} \in G = C_n \odot K_m. \end{cases}$$

Theorem 2.2: A function $f: E \to \{-1, 1\}$ defined by

$$f(e) = \begin{cases} -1, \text{for } (m-1) \text{ edges } e = l_{ij} \text{ in each copy of } K_m, \\ 1, \text{ otherwise.} \end{cases}$$

is a minimal signed edge dominating function of $G = C_n \odot K_m$.

Proof: Let f be a function defined as in the hypothesis. By the definition of the function - 1 is assigned to $(m-1)$ edges l_{ij} in *each* copy of K_m in G and 1 is signed to the remaining edges of G. The summation value taken over $\mathrm{N}[e]$ of $e \in E$ is as follows.

Case 1: Let $e_i \in C_n$, be such that $adj(e_i) = 2m+2$ in G.

Then $\mathrm{N}[e_i]$ contains three edges of C_n and $2m$ edges which are drawn from the vertices v_i and v_{i+1} respectively to the m vertices of i^{th} and $(i+1)^{th}$ copies of K_m and their functional value is 1 .

$$\text{Therefore } \sum_{e \in N[e_i]} f(e) = 1 + 1 + 1 + \underbrace{[1 + 1 + \cdots + 1]}_{2m-times} = 2m + 3.$$

Case 2: Let $l_{ik} \in i^{th}$ copy of K_m. By the definition of f, $(m-1)$ edges l_{ij} are assigned -1 and the remaining edges are assigned 1.

By Theorem 2.1, $adj(l_{ik}) = 2m-2$. That is the edge l_{ik} is adjacent to $(2m-4)$ edges l_{ij}, $j \neq k$ and two edges h_{ij}. Here $f(h_{ij}) = 1$.

$$\text{So } \sum_{e \in N[l_{ik}]} f(e) = [(m-1)(-1) + (m-2)(1)] + 1 + 1 = 1.$$

Now for all other possibilities of functional values of l_{ij} that are adjacent to l_{ik}, $k = 1, 2, \ldots, m$, we could see that

$$\sum_{e \in N[l_{ij}]} f(e) > 1.$$

Case 3: Let $h_{ij} \in C_n \odot K_m$ be such that $adj(h_{ij}) = 2m$ in G.

Then $N[h_{ij}]$ contains two edges of C_n , m edges h_{ij} and $(m-1)$ edges l_{ij} in K_m.

Suppose $f(l_{ij}) = -1$ for all $(m-1)$ edges l_{ij} that are adjacent to h_{ij}. Then

$$\sum_{e \,\epsilon\, N[h_{ij}]} f(e) = 1 + 1 + [(m-1)(-1) + (m)(1)]$$
$$= 3.$$

Suppose $f(l_{ij}) = 1$ for all $(m-1)$ edges l_{ij} that are adjacent to h_{ij}. Then

$$\sum_{e \,\epsilon\, N[h_{ij}]} f(e) = 1 + 1 + [(m-1)(1) + (m)(1)]$$
$$= 2m + 1.$$

Thus as in Case 2 for all other possibilities of functional values for the $(m-1)$ edges that are adjacent to h_{ij} we could see that

$$\sum_{e \in N[h_{ij}]} f(e) > 1.$$

Therefore, for all possibilities we get

$$\sum_{e \in E(G)} f(e) \geq 1, \text{for all } e \in E(G).$$

Hence f is a signed edge dominating function.

We now check for the minimality of f.

Define a function $g : E \rightarrow \{-1, 1\}$ by

$$g(e) = \begin{cases} -1, \text{for one edge } h_{ik}, \\ -1, \text{for } (m-1) \text{ edges } l_{ij} \text{ in each copy of } K_m, \\ \ \ 1, \text{otherwise.} \end{cases}$$

Since strict inequality holds at h_{ik} it follows that $g < f$.

Case (i): Let $e_i \in C_n$ be such that $adj(e_i) = 2m + 2$ in G.

Sub Case 1: Let $h_{ik} \in N[e_i]$. Then

$$\sum_{e \,\epsilon\, N[e_i]} g(e) = -1 + 1 + 1 + \underbrace{[1 + 1 + \cdots + 1]}_{2m-times}$$
$$= 2m + 1.$$

Sub Case 2: Let $h_{ik} \notin N[e_i]$. Then

$$\sum_{e \,\epsilon\, N[e_i]} g(e) = 1 + 1 + 1 + \underbrace{[1 + 1 + \cdots + 1]}_{2m-times}$$
$$= 2m + 3.$$

Case (ii): Let $l_{ik} \in i^{th}$ copy of K_m. Then $adj(l_{ik}) = 2m - 2$ in G.

Sub Case 1: Let $h_{ik} \in N[l_{ik}]$. Then

$$\sum_{e \,\epsilon\, N[l_{ik}]} g(e) = [(m-1)(-1) + (m-1)(1)] + (-1) = -1.$$

Sub Case 2: Let $h_{ik} \notin N[l_{ik}]$. Then

$$\sum_{e \,\epsilon\, N[l_{ik}]} g(e) = [(m-1)(-1) + (m-1)(1)] + 1$$
$$= 1.$$

Now for all other possibilities of functional values of l_{ij} that are adjacent to l_{ik} we could see that

$$\sum_{e \in N[l_{ij}]} g(e) > 1.$$

Case (iii): Let $h_{ij} \in C_n \odot K_m$ be such that $adj(h_{ij}) = 2m$ in G.

Sub case 1: Let $h_{ik} \in N[h_{ij}]$.

Suppose $g(l_{ij}) = -1$ for all $(m-1)$ edges l_{ij} that are adjacent to h_{ij}. Then

$$\sum_{\boldsymbol{e \,\epsilon\, N[h_{ij}]}} \boldsymbol{g(e) = 1 + 1 + [(m-1)(-1) + (m-1)(1)] + (-1) = 1}.$$

Suppose $\boldsymbol{g(l_{ij}) = 1}$ for all $\boldsymbol{(m-1)}$ edges $\boldsymbol{l_{ij}}$ that are adjacent to $\boldsymbol{h_{ij}}$. Then

$$\sum_{e \in N[h_{ij}]} g(e) = 1 + 1 + [(m-1)(1) + (m-1)(1)] + (-1) = 2m - 1.$$

Sub Case 2: Let $h_{ik} \notin N[h_{ij}]$.

Suppose $g(l_{ij}) = -1$ for all $(m-1)$ edges l_{ij} that are adjacent to h_{ij}. Then

$$\sum_{e \in N[h_{ij}]} g(e) = 1 + 1 + [(m-1)(-1) + (m)(1)] = 3.$$

Suppose $g(l_{ij}) = 1$ for all $(m-1)$ edges l_{ij} that are adjacent to h_{ij}. Then

$$\sum_{e \in N[h_{ij}]} g(e) = 1 + 1 + [(m-1)(1) + (m)(1)] = 2m + 1.$$

We see that

$$\sum_{e \in E[G]} g(e) < 1 \text{, for some } e \in E(G).$$

So g is not an edge dominating function. Since g is defined arbitrarily, it follows that there exists no $g < f$ such that g is an edge dominating function.

Thus f is a minimal signed edge dominating function. ■

III. ROMAN EDGE DOMINATING FUNCTION

The Roman dominating function of a graph G was defined by Cockayne et al. [8]. The definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart [15] entitled "Defend the Roman Empire!", Henning et.al [12] and suggested even earlier by ReVelle [19].

In this section first we recall the definitions of Roman edge dominating function of a graph. Later results on minimal Roman edge dominating functions of $G = C_n \odot K_m$ are discussed.

Definition: Let $G(V, E)$ be a graph. A function $f : E \to \{0, 1, 2\}$ is called a Roman edge dominating function (REDF) of G if

$$f(\mathrm{N}[e]) = \sum_{e' \in \mathrm{N}[e]} f(e') \geq 1, \forall e \in E(G)$$

and satisfying the condition that every edge e for which $f(e) = 0$ is adjacent to at least one edge e' for which $f(e') = 2$.

A Roman edge dominating function f of G is called a minimal Roman edge dominating function (MREDF) if for all $g < f$, g is not a Roman edge dominating function.

Theorem 3.1: A function $f : E \to \{0, 1, 2\}$ defined by

$$f(e) = \begin{cases} 2, \text{ for } (m-1) \text{ edges } h_{ij} \text{ in } C_n \odot K_m, \\ 0, \text{ otherwise.} \end{cases}$$

is a minimal Roman edge dominating function of $G = C_n \odot K_m$.

Proof: Let f be a function defined as in the hypothesis.

Case 1: Let $e_i \in C_n$ be such that $adj(e_i) = 2m + 2$ in G.

Then $N[e_i]$ contains m edges $h_{i1}, h_{i2}, \ldots, h_{im}$ of G ,again m edges $h_{(i+1)1}, h_{(i+1)2}, \ldots, h_{(i+1)m}$ of G and three edges of C_n.

$$\text{So} \sum_{e \in N[e_i]} f(e) = 0 + 0 + 0 + \underbrace{[2 + 2 + \cdots + 2]}_{(2m-2)\, times} + 0 + 0 = 4m - 4.$$

Case 2: Let $l_{ij} \in i^{th}$ copy of K_m. Then $adj(l_{ij}) = 2m - 2$ in G .

Then $N[l_{ij}]$ contains $(2m-3)$ edges of K_m and two edges h_{ij} of G. Then

$$\sum_{e \in N[l_{ij}]} f(e) = \underbrace{[0 + 0 + \cdots + 0]}_{(2m-3)-times} + 2 + 2 = 4.$$

or

$$\sum_{e\,\epsilon\,N[l_{ij}]} f(e) = \underbrace{[0+0+\cdots+0]}_{(2m-3)-times} + 2 + 0 = 2.$$

Case 3: Let $h_{ij} \in C_n \odot K_m$ be such that $adj(h_{ij}) = 2m$ in G.

$$\textbf{Then} \sum_{e\,\epsilon N[h_{ij}]} f(e) = 0 + 0 + [(m-1)2 + (m-1)0] + 0 = 2m - 2.$$

Therefore for all possibilities, we get

$$\sum_{e \in E[G]} f(e) > 1.$$

Let e be an edge of G such that $f(e) = 0$ and e' be another edge of G such that $e' \neq e$ and $f(e') = 2$. Then we show that e and e' are adjacent.

Now $f(e) = 0$ implies $e = e_i \in C_n$ for some i, or $e = l_{ij}$, for some $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Now $f(e') = 2$ implies $e' = h_{ij}$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, (m-1)$.

Suppose $e = e_i \in C_n$. Then obviously e_i and h_{ij} are adjacent. That is e and e' are adjacent.

Suppose $e = l_{ij}$ for some i and j. Then l_{ij} and h_{ij} are adjacent. That is e and e' are adjacent.

This implies that f is a Roman edge dominating function. Now we check for the minimality of f. Define a function $g : E \rightarrow \{0, 1, 2\}$ by

$$g(e) = \begin{cases} \mathbf{1, for\ one\ edge}\ h_{ik}\ \mathbf{in}\ C_n \odot K_m, \\ \mathbf{2, for}\ (\mathbf{m}-\mathbf{2})\ \mathbf{edges}\ h_{ij}\ \mathbf{in}\ C_n \odot K_m,\ \mathbf{j} \neq \mathbf{k}, \\ \mathbf{0, otherwise.} \end{cases}$$

Since strict inequality holds at an edge h_{ik}, it follows that $g < f$.

Case (i): Let $e_i \in C_n$ be such that $adj(e_i) = 2m + 2$ in G.

Sub Case 1: Let $h_{ik} \in N[e_i]$. Then

$$\sum_{e\,\epsilon\,N[e_i]} g(e) = 0 + 0 + 0 + \underbrace{[2+2+\cdots+2]}_{(2m-3)times} + 1 + 0 + 0 = 4m - 5.$$

Sub Case 2: Let $h_{ik} \notin N[e_i]$. Then

$$\sum_{e\,\epsilon\,N[e_i]} g(e) = 0 + 0 + 0 + \underbrace{[2+2+\cdots+2]}_{(2m-2)times} + 0 + 0 = 4m - 4.$$

Case (ii): Let $l_{ij} \in i^{th}$ copy of K_m. Then $adj(l_{ij}) = 2m - 2$ in G.

Sub Case 1: Let $h_{ik} \in N[l_{ij}]$. Then

$$\sum_{e\,\epsilon\,N[l_{ij}]} g(e) = \underbrace{[0+0+\cdots+0]}_{(2m-3)times} + 1 + 2 = 3,$$

or

$$\sum_{e\,\epsilon\,N[l_{ij}]} g(e) = \underbrace{[0+0+\cdots+0]}_{(2m-3)times} + 1 + 0 = 1.$$

Sub Case 2: Let $h_{ik} \notin N[l_{ij}]$. Then

$$\sum_{e\,\epsilon\,N[l_{ij}]} g(e) = \underbrace{[0+0+\cdots+0]}_{(2m-3)times} + 2 + 2 = 4,$$

or

$$\sum_{e\,\epsilon\,N[l_{ij}]} g(e) = \underbrace{[0+0+\cdots+0]}_{(2m-3)times} + 2 + 0 = 2.$$

Case (iii): Let $h_{ij} \in C_n \odot K_m$ be such that $adj(h_{ij}) = 2m$ in G.

Sub Case 1: Let $h_{ik} \in N[h_{ij}]$. Then

$$\sum_{e\,\epsilon N[h_{ij}]} g(e) = 0 + 0 + [(m-2)2 + m(0)] + 1 = 2m - 3.$$

Sub Case 2: Let $h_{ik} \notin N[h_{ij}]$.

$$\sum_{e \,\epsilon N[h_{ij}]} g(e) = 0 + 0 + [(m-1)2 + m(0)]$$
$$= 2m - 2.$$

Hence for all possibilities, we get

$$\sum_{e \epsilon E(G)} g(e) > 1 \text{, for all } e \in E(G).$$

i.e. g is an edge dominating function. But g is not a Roman edge dominating function, since the REDF definition fails in the i^{th} copy of K_m in G.

Let the edge $l_{ij} \in i^{th}$ copy of K_m. Then $g(l_{ij}) = 0$. We know that every edge l_{ij} in K_m is adjacent to two edges $h_{ij}, j = 1, 2, \dots, m$.

The condition of Roman dominating function fails for the edge l_{ij} which is adjacent to h_{ik} and h_{ij} where $g(h_{ik}) = 1$ and $g(h_{ij}) = 0$.

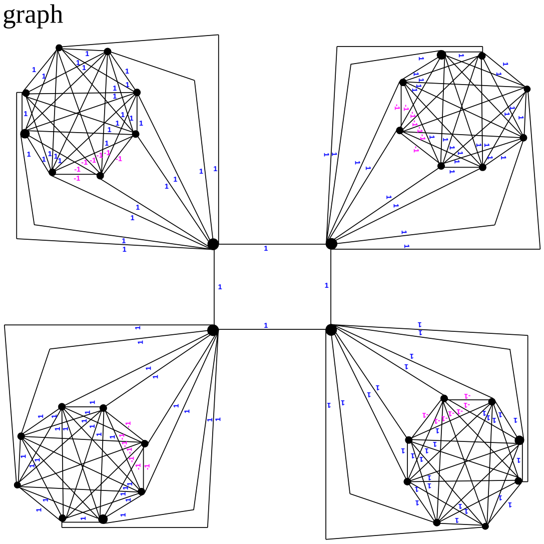
Thus f is a minimal Roman edge dominating function. ■

IV. ILLUSTRATIONS

4.1 Minimal Signed Edge Dominating Function

Theorem 2.2

The functional values are given at each edge of the graph G.

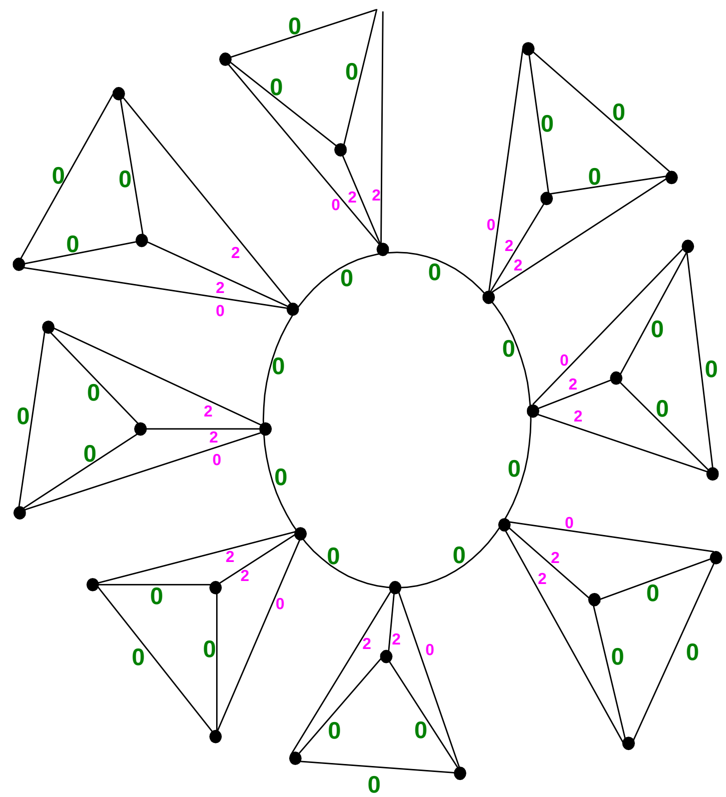


$G = C_4 \odot K_8$

4.2 Minimal Roman Edge Dominating Function

Theorem 3.1

The functional values are given at each edge of the graph G.



$G = C_8 \odot K_3$

REFERENCES

[1] Allan, R.B. and Laskar, R.C.– On domination, independent domination numbers of a graph, Discrete Math.23, 1978, pp.73 –76

[2] Anitha, J and Maheswari, B. - Edge Dominating functions of Corona Product Graph of a Cycle with a Complete Graph- Open journal of Applied and Theoretical Mathematics (OJATM), ¬Volume.2, No-4, December (2016), pp.151-160.

[3] Anitha, J. and Maheswari, B. - Edge Dominating number of Corona Product Graph of a Cycle with a Complete Graph- International Journal of Computer Applications (IJCA), Volume 158, No 3, January (2017), pp. 40-44.

[4] Anitha, J. and Maheswari, B. - Total Edge Dominating sets and functions of Corona Product Graph of a Cycle with a Complete Graph – Asia Mathematika (Communicated).
[5] Arumugam S., Sithara Jerry. - Fractional edge domination in graphs, Appl. Anal. Discrete math.3, 2009, pp.359-370.
[6] Arumugam S., Velammal S. - Edge domination in graphs, Taiwanese Journal of Mathematics, 2 (2), 1998, pp.173-179.
[7] Cockayne, E.J. and Hedetniemi, S.T. - Towards a theory of domination in graphs, Networks, 7, 1977, pp.247 – 261.
[8] Cockayne, E.J.,Dreyer, P.A., Hedetniemi, S.M. and Hedetniemi, S.T. - Roman domination in graphs, Discrete Math., 278 (2004), pp.11 – 22.
[9] Dunbar.J., Hedetniemi.S.T., Henning, M.A. and Slater, P.J. - Signed domination in graphs, in: Y. Alari and A. Schwenk (Eds.), Proc. 7thInternat. Conf. on the Theory and Applications of Graphs, Wiley, New York, 1995, pp. 311 – 321.
[10] R. Dutton and W. F. Klostermeyer - Edge dominating sets and vertex covers, Discussions Mathematicae, vol. 33, no.2, 2013, pp.437-456.
[11] Frucht, R. and Harary, F. - On the corona of Two Graphs, Aequationes Mathematicae, Volume 4, Issue 3, 1970, pp.322 – 325.
[12] Henning, M.A. and Hedetniemi, S.M – Defending the Roman Empire – A new strategy, Discrete Mathematics, 266 (2003), pp.239 – 251.
[13] Haynes, T.W., Hedetniemi, S.T. and Slater, P.J. -Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., New York, 1998.
[14] Haynes, T.W., Hedetniemi, S.T. and Slater, P.J. - Fundamentals of domination in graphs, Marcel Dekker, Inc., New York 1998.
[15] Ian Stewart - Defend the Roman Empire!, Scientific American, 281(6), 1999, pp.136 – 139.
[16] Jayaram, S. R. - Line domination in graphs, Graphs and Combinatorics, vol. 3, no. 4, 1987, pp. 357–363.
[17] Kulli, R., Soner, N. D. - Complementary edge domination in graphs, Indian Journal of Pure and Applied Mathematics, vol. 28, no. 7, 1997,pp. 917–920.
[18] Mitchell S, Hedetniemi, S.T. - Edge domination in trees. Congr. Numer., 19 , 1977 , pp.489-509.
[19] ReVelle, C.S. and Rosing, K.E. - Defendens imperium romanum: a classical problem in military strategy, Amer. Math. Monthly, 107 (7) (2000), pp.585 – 594.
[20] Yannakakis, M., Gavril, F. - Edge dominating sets in graphs, SIAM Journal on Applied Mathematics, vol. 38, no. 3, 1980, pp. 364–372.