Rules of Numerical Integration in Details

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Abstract -- In this review paper we have tried to write all the points related to the numerical integration. Numerical integration are not the normals integration formulas but they are solved by writing different formulas which are not the general integration formulas. They are simply the algebraic formulas which can be solved easily, although they are quite leandy. There are three ways to solve the definite integrals (in which limits are given) -

1. ***Trapezoidal Rule***
2. ***Simpson's 1/3rd Rule***
3. ***Simpson's 3/8th Rule***

In this paper we have tried our best for the explanation of all these three topics and also have given example for further understanding.

I. INTRODUCTION

Integration is the method of finding the area of the function plot on the graph.

Trapezoidal rule - A trapezoidal rule is done by the approximation of the region which is covered by the graph of the function f(x) as a trapeziod and further measuring its area. In this the interval is broken into subintervals and then chained as a sum to get the appropriate definite integral answer. It is based on the approximation of integral by a first order polynomial [6].

Its equation is given as -
$I = h/2\,[\,(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1})]$

Simpson's 1/3rd rule – It is similar to the trapezoidal rule which is used to calculate the area of a given function by the definite integral form. Only the difference is that it is the extended form of trapezoidal rule as it is used to calculate the second order polynomial [1].

Its equation is given as -
$I = h/3\,[\,(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots\ldots.) + 2(y_2 + y_4 + y_6 + \ldots\ldots.)]$

Simpson's 3/8th rule - Simpson's 3/8 rule is designed to calculate the third order (cubic) polynomial function.
Its equation is given as -
$I = 3h/8[\,(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \ldots..) + 2(y_3 + y_6 + y_9 + \ldots\ldots\ldots.)]$

Trapezoidal Rule - Let the function be given f(x) whose integration we have to calculate from x_0 to x_n . The function is given as y = f(x).
Now, the function according to the limits are given as -
$Y_0 = f(x_0)$ and
$Y_n = f(x_n)$.
$f(x) = f(x_0) + f(x_n)$

I (whose limit is from x_0 to x_n) = $\int [\ f(x_0) + \{\ (f(x_n) - f(x_0))\ (x - x_0)\ \} / (x_n - x_0)\ ]\ dx$

I (whose limit is from x_0 to x_n) = $\int [\ f(x_0) + \{\ xf(x_n) - xf(x_0)\ \} / (x_n - x_o)\ - \{\ f(x_n)x_0 - f(x_0)x_0\ \} / (\ x_n - x_0\ )\ ]\ dx$

I (whose limit is from x_0 to x_n) = $\int [\ \{\ x_nf(x_0) - x_0f(x_0) + xf(x_n) - xf(x_0) - x_0f(x_n) + x_0f(x_0)\ \} / (x_n - x_0)\ ]\ dx$

I (whose limit is from x_0 to x_n) = $\int [\ \{\ x_nf(x_0) + xf(x_n) - xf(x_0) - x_0f(x_n)\ \} / (x_n - x_0)\ ]\ dx$

I (putting both upper and lower limits) = $x_nf(x_0) - x_0 f(x_n) + \{\ (x_n + x_0)(f(x_n) - f(x_0))\} / 2$

$I = \{2x_n\, f(x_0) - 2x_0f(x_n) + x_nf(x_n) - x_nf(x_0) + x_0f(x_n) - x_0f(x_0)\ \} / 2$

$I = \{ x_nf(x_0) – x_0f(x_n) + x_nf(x_n) – x_0f(x_0) \} / 2$

$I = \{ x_n (f(x_0) + f(x_n)) – x_0(f(x_n) + f(x_0)) \} / 2$

$I = (x_n – x_0)(f(x_n) + f(x_0)) / 2$

$I = (x_n – x_0)(y_n + y_0) / 2$

I = width * average of y coordinate

Composite Integral –
$\int(x_0 \text{ to } x_n)\ y\ dx = \int(x_0 \text{ to } x_1)\ y\ dx + \int(x_1 \text{ to } x_2)\ y\ dx + \int(x_2 \text{ to } x_3)\ y\ dx + \ldots\ldots$

$I = [(x_1 – x_0)(y_1 + y_0)/2 + (x_2 – x_1)(y_2 + y_1)/2 + (x_3 – x_2)(y_3 + y_2)/2 + \ldots\ldots.]$

$\{ \text{Since } (x_1 – x_0) = (x_2 – x_1) = (x_3 – x_2) = h \}$

$I = \int(x_0 \text{ to } x_n)\ y\ dx = h [(y_1 + y_0)/2 + (y_2 + y_1)/2 + (y_3 + y_2)/2 + \ldots\ldots\ldots\ldots]$

$I = h [y_0/2 + y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots + y_n/2]$

$I = h/2 [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1})]$

Is the required trapezoidal rule.

SIMPSON'S 1/3rd RULE –
It is the 3 points and 2nd degreee equation.

It has three points $A(x_0 , y_0)$, $B(x_k , y_k)$, $C(x_n , y_n)$

Let the equation of parabola is given as y = f(x)[5].
Now,

By lagrange's interpolation method-

$Y = f(x) = \{ (x – x_k)(x – x_n)/(x_0 – x_k)(x_0 – x_n) \} * y_0 + \{ (x – x_0)(x – x_n) / (x_k – x_0)(x_k – x_n) \} * y_k + \{ (x – x_0)(x – x_k)/(x_n – x_0)(x_n – x_k) \} * y_n$

$\int(x_0 \text{ to } x_n)\ ydx = \int(x_0 \text{ to } x_n) [\{ (x^2 – xx_n – xx_k + x_nx_k)/(x_0 – x_k)(x_0 – x_n) \} * y_0 + \{ (x^2 – xx_n – xx_0 + x_0x_n)/(x_k – x_0)(x_k – x_n) \} * y_k + \{ (x^2 – xx_k – xx_0 – x_0x_k)/(x_n – x_0)(x_n – x_k) \} * y_n] dx$

$\int(x_0 \text{ to } x_n)\ ydx = \int(x_0 \text{ to } x_n) [\{ (x^2 – (x_n + x_k)x + x_nx_k)/(x_0 – x_k)(x_0 – x_n) \} * y_0 + \{ (x^2 – (x_n + x_0)x + x_0 x_n)/(x_k – x_0)(x_k – x_n) \} * y_k + \{ (x^2 – (x_k + x_0)x + x_0 x_k)/(x_n – x_0)(x_n – x_k) \} * y_n] dx$

$I = y_0 /(x_0 – x_k)(x_0 – x_n)[(x_n^3 – x_0^3)/3 – (x_n + x_k)(x_n^2 – x_0^2)/2 + x_nx_k(x_n – x_0)] + y_k /(x_k – x_0)(x_k – x_n) [(x_n^3 – x_0^3)/3 – (x_n + x_0)(x_n^2 – x_0^2)/2 + x_0 x_n(x_n – x_0)] + y_n /(x_n – x_0)(x_n – x_k) [(x_n^3 – x_0^3)/3 – (x_k + x_0)(x_n^2 – x_0^2)/2 + x_0 x_k(x_n – x_0)]$

$\int(x_0 \text{ to } x_n)\ y\ dx = (x_n – x_0)/6 [y_0 + 4y_k + 2y_n]$
$= h/3(y_0 + 4y_k + 2y_n)$
$= h/3(y_0 + 4y_1 + 2y_2) + h/3(y_2 + 4y_3 + 2y_4) + h/3(y_4 + 4y_5 + 2y_6)$

$I = h/3 [(y_0 +y_n) + 4(y_1 + y_3 + y_5 + \ldots\ldots.) + 2(y_2 + y_4 + y_6 + \ldots\ldots.)]$

Therefore, this is the required simpson's 1/3rd rule.

SIMPSON'S 3/8th RULE –

Taking four points (x_0,y_0) , (x_1,y_1) , (x_2,y_2) and (x_3,y_3)
Now, finding the integration of function y from x_0 to x_3 we get-
$\int(x_0 \text{ to } x_3)\ y\ dx =[\{ (x – x_1)(x – x_2)(x – x_3)/(x_0 – x_1)(x_0 – x_2)(x_0 – x_3) \}* y_0 + \{ (x – x_0)(x – x_2)(x – x_3)/(x_1 – x_0)(x_1 – x_2)(x_1 – x_3) \}* y_1 + \{ (x – x_0)(x – x_1)(x – x_3)/(x_2 – x_0)(x_2 – x_1)(x_2 – x_3) \}* y_2 + \{ (x – x_0)(x – x_1)(x – x_2)/(x_3 – x_0)(x_3 – x_1)(x_3 – x_2) \}* y_3] dx$

On solving this equation we get-
$I = (x_3 – x_0)/8 (y_0 + 3y_1 + 3y_2 + 2y_3)$

Here we know that $h = x_1 – x_0 = x_2 – x_1 = x_3 – x_2$
Therefore,

$\int(x_0 \text{ to } x_n)\ y\ dx = 3h/8 * [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 +\ldots..) + 2(y_3 + y_6 + y_9 + \ldots\ldots\ldots.)]$

Is the required simpson's 3/8th rule.

EXAMPLES:
Que. 01. Evaluate integration $\int(0 \text{ to } 0.9) \log_e(1 + \sqrt{x})dx$ using trapezoidal rule with 9 subinterval.

Ans. Here $x_0 = 0$, $x_n = 0.9$ and $n = 9$
Now, $h = (0.9 - 0)/9$
$= 0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
y	0	0.274	0.369	0.436	0.490	0.534	0.573	0.607	0.638	0.667

Now by trapezoidal rule-
I = 0.1/2 [(0 + 0.667) + 2(0.274 + 0.369 + 0.436 + 0.490 + 0.534 + 0.573 + 0.607 + 0.638)]

Therefore I = 0.42545 is the integration.

Que. 02. Evaluate ∫(1.0 to 1.8) $(e^{x} + e^{-x})/2$ dx using simpson's $1/3^{rd}$ rule
By taking h = 0.2.
Ans. Here $x_0 = 1.0$, $x_n = 1.8$ and h = 0.2
Now, n = (1.8 – 1.0)/ 0.2 = 4

X	1.0	1.2	1.4	1.6	1.8
y	1.5430	1.8106	2.1508	2.5774	3.1074

Now by simpson's $1/3^{rd}$ rule -
I = 0.2/3 [(1.5430 + 3.1074) + 4(1.8106 + 2.5774) + 2(2.1508)]

Therefore I = 1.764 is the integration.

Que. 03. Evaluate ∫(0 to π/2) $e^{\sin x}$ dx using simpson's $3/8^{th}$ rule with 6 subintervals.

Ans. Here, $x_0 = 0$, $x_n = \pi/2$ and n = 6
Now, h = (π/2 - 0) / 6
= π/12

X	0	π/12	π/6	π/4	π/3	5π/12	π/2
Y	1	1.2951	1.6483	2.0275	2.3768	2.6267	2.7182

Now by simpson's $3/8^{th}$ rule-
I = 3π/12*8 [(1 + 2.7182) + 3(1.2951 + 1.6483 + 2.3768 + 2.6267) + 2 * 2.0275]

Therefore I = 3.1021 is the integration.

II. APPLICATIONS

- By these method we can find extremely accurate answer when the function is periodic i.e. the interval is periodic.
- By this method there is faster convergence of the function (i.e. they can be calculated faster without the remembrance of the formula).

III. CONCLUSION

By this paper we come to known about the different ways which are helpful to calculate the numerical integrals, their formulas and the applications. Examples are also helpful to understand the topic.

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