Consecutive Cyclic k – Prime Labeling Of a Ladder Graphand Its Application to Crab Graph

M.D.M.C.P. WEERARATHNA[1], D. M. T. B. DISSANAYAKE2, W. V. NISHADI[3], K.D.E. DHANANJAYA[4], T.R.D.S.M. THENNAKOON[5], A. A. I. PERERA[6]
[1, 2, 3, 4, 5, 6] *Department of Mathematics, Faculty of Science, University of Peradeniya, Sri Lanka*.

Abstract- A graph G order n is said to have a k−prime labeling, where$k$ is a positive integer, if the vertices can be labeled with positive integers from k to $k+|n|-1$ such that each pair of adjacent vertices are relatively prime. In this work, a consecutive cyclic k – prime labeling for ladder graphs is given under certain conditions. Furthermore, a new graph called Crab graph is introduced and a prime labeling is discussed for it.

Indexed Terms- Crab Graph, Ladder Graph, k-prime labeling, prime labeling.

I. INTRODUCTION

Graph labeling is an active research area in Graph theory due to the numerous number of open problems and vast amount of literature is available on graph labeling for different types of graphs. Prime labeling is a special case in graph labeling.

A *Prime labeling* of a simplegraph$G = G(V, E)$, where V and E denotes the set of vertices and edges respectively is a labeling of vertices of G with distinct integers from the set $\{1, 2, \cdots, n\}$ in such a way that the labels of any two adjacent vertices are relatively prime. Such a graph is called a *Prime graph*. Two integers are said to be relatively prime, if their greatest common divisor (gcd) is 1.

The concept of prime labeling was introduced by Roger Entringer around 1980's where he stated the conjecture; every tree is prime, which is remained unsolved. The theory was developed and discussed by A. Tout et al. in 1982 [4]. Recent work on prime labeling involves for known graphs. In 2011, Vaidya and Prajapati introduced the concept of k−prime labeling and investigated some results concerning to it [6].

In our previous researches prime labeling for Tripartite graphs, Roach graph and Scorpion graph have been introduced. Present work focuses on k – prime labeling for ladder graphs which is analogous to those stated by A. H. Berlineer et al. in [2], where they considered prime and coprime labeling for ladder graphs.

Furthermore, in this paper, we introduce a new graph named as Crab graph which shows the shape of a crab. This graph is introduced considering the real shape of a crab. Following figure illustrates the Crab graph.

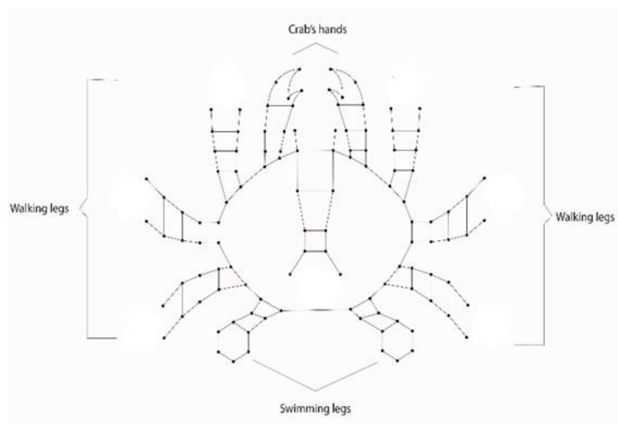


Figure 1: Crab graph

A prime labeling is given to this graph using the result obtained for k-prime labeling of ladder graph.

Some important and useful definitions for the present investigation are stated below.

Definition ($k-$ prime labeling)

A k – prime labeling of a graph $G = (V, E)$ with the vertex set V and edge set E is an injective function $f: V \longrightarrow \{k, k+1, k+2, \ldots, k+|V|-1\}$ for some positive integer k that induces a function $f^{+}: E(G) \longrightarrow$

$\mathbb{N}$ of the edges of G defined by $f^+(uv) = \gcd(f(u), f(v)), \forall\, e = uv \in E(G)$ such that $\gcd(f(u), f(v)) = 1$. A graph G that admits k – prime labeling is called a k – prime graph.[6]

Definition (Ladder Graph)

If P_n denote the path on n vertices, then the Cartesian product $P_n \times P_m$, where $m \leq n$, is called a *grid graph.* If $m = 2$, then the graph is called a *ladder*.[2]

II. METHODOLOGY

Theorem

A graph of the form $P_n \times P_2$ has a consecutive cyclic k – prime labeling with any integer $x(\geq 1)$ assigned to the top left vertex u_1 if and only if $2(x+n) - 1$ is prime ($n \in \mathbb{Z}^+$).

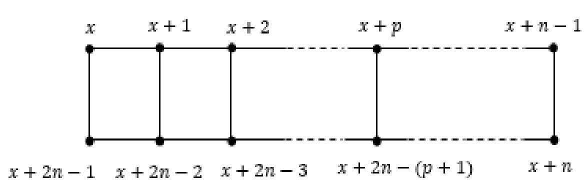
Proof:

The forward implication proved by contrapositive statement.

Let any $p \in \mathbb{Z}^+$ and assume that $x + p > 1$ for $p \geq 0$ be a divisor of $2(x+n) - 1$.

i.e. $(x+p)/[2(x+p) - 1]$

Consider the following consecutive cyclic labeling $P_n \times P_2$ with the value of x assigned to the top left vertex (say u_1) of the graph,



Generally,

The pair of vertices labeled $x + p$ and $x + 2n - (p+1) = (2x + 2n - 1) - (x + p)$ are not relatively prime since, $(x+p)/(2x + 2n - 1)$.

Hence, the above graph is not prime labeling.

Therefore, if a graph of the form $P_n \times P_2$ has a consecutive cyclic k – prime labeling with any integer $x(\geq 1)$ assigned to the top left vertex u_1, then $2(x+n) - 1$ is prime. ($n \in \mathbb{Z}^+$)

Conversely,

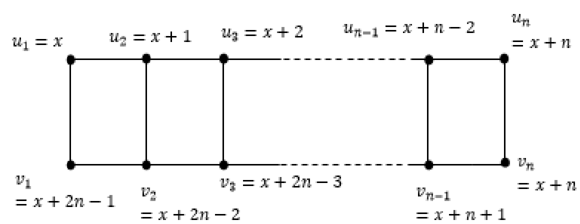
Consider the vertex labeling of ladder graph $P_n \times P_2$ such that $2(x+n) - 1$ is prime.

Claim:

Let $P_n \times P_2$ be the ladder with top vertices $u_1, u_2, \ldots, u_n$, and bottom vertices $v_1, v_2, \ldots, v_n$, where u_i is adjacent to v_i for $1 \leq i \leq n$, u_i is adjacent to u_{i+1} for $1 \leq i \leq n-1$ and v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$.

A consecutive cyclic prime labeling of a ladder $P_n \times P_2$ is a prime labeling in which the labels on the vertices wrap around the ladder in a consecutive way.

The following vertex labeling gives a consecutive cyclic prime labeling.



Since, $\gcd(k, k+1) = 1$ for $k > 0$ (i.e. consecutive positive numbers are prime),

It suffices to check only the vertex labels arising from the end points of first $(n-1)$ vertical edges going from left to right.

Observe that each of the $(n-1)$ vertical edges under consideration have vertex labels $(x+p)$ and $[2(x+n) - 1] - (x+p)$ for $1 \leq x < x + p \leq n - 1$and $p > 0, p, x \in \mathbb{Z}^+$.

Then,

$\gcd(x+p, [2(x+n) - 1] - (x+p)) = \gcd(x+p, 2(x+n) - 1) = 1$, since $2x + 2n - 1$ is prime.

Thus, the graph $P_n \times P_n$ has a consecutive cyclic k – prime labeling whenever $2(x+n)-1$ is prime.

III. RESULTS AND DISCUSSION

The above theorem can be used to give a prime labeling for the Crab graph.

First label the middle partof the shell, shown by ⑦ in figure 2 using cyclic prime labeling method of the ladder graph.

i.e. There are 6 vertices including crab's eye in a one side of ⑦.

Therefore, $n = 6$ and $2n + 1 = 12 + 1 = 13$ which is a prime. Since this part is similar to ladder graph and $2n + 1$ is prime, this part has a cyclic prime labeling.

Starting right side eye(u_1) as a $u_1 = 1$ and it's ending with left eye $v_1 = 12$.

Then, $\gcd\big(f(u_i), f(v_i)\big) = 1$, where,$u_i$ = right side vertical vertices and v_i = left side vertical vertices for $1 \le i \le 6$ (by cyclic prime labeling theorem)

$\gcd\big(f(u_i), f(u_{i+1})\big) = 1$ (consecutive positive numbers) for $1 \le i \le 5$

$\gcd\big(f(v_i), f(v_{i+1})\big) = 1$ (consecutive positive numbers) for $1 \le i \le 5$

Therefore, part labeled by ⑦ has a prime labeling.

Now, consider crab's right hand.

Let $u_7 = 13$

With the vertices u_{11} and v_{11} joined, this right hand is isomorphic to the ladder graph $P_6 \times P_2$.

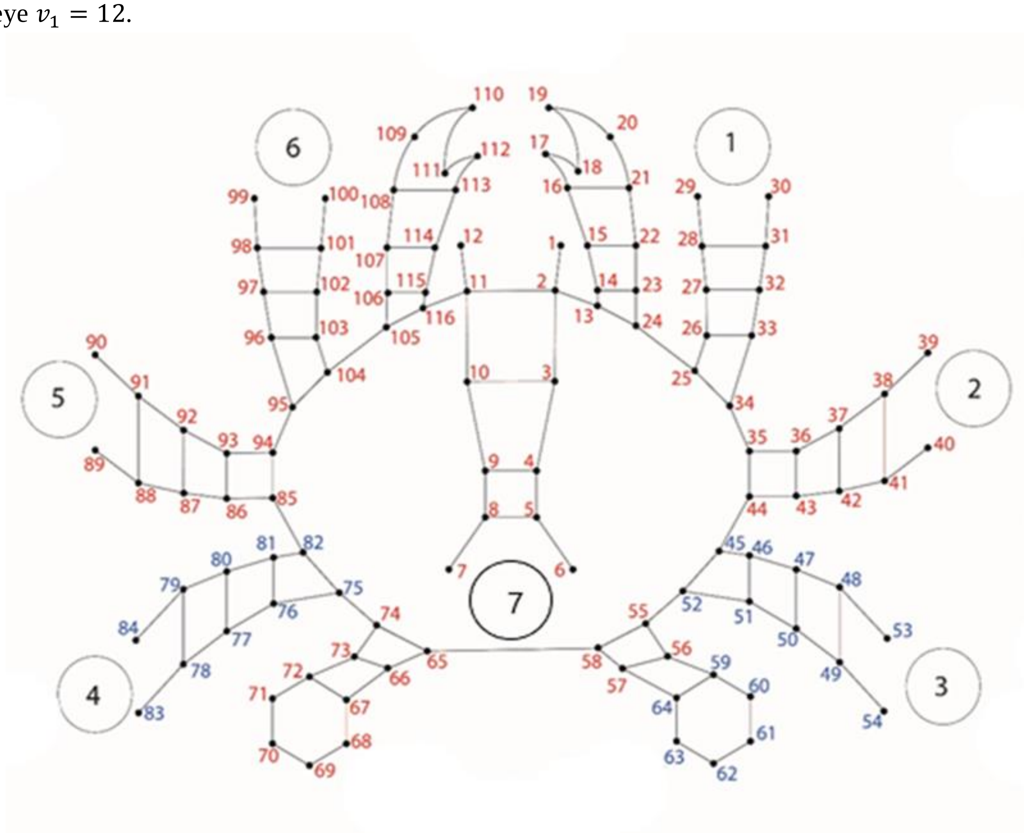


Figure 2: Prime Labeling of Crab graph

Then, there are 6 pair of vertices and $n = 6$ and let $u_7 = x = 13$.

Consider, $2(x+n)-1 = 2(13+6)-1 = 37$ and this is a prime number.

Therefore, consecutive cyclic k – prime labeling theorem can be used to show that right hand has a k – prime labeling starting with $u_7 = 13$ and ending with $v_7 = x + 2n - 1 = 13 + 12 - 1 = 24$.

Then, $\gcd\big(f(u_i), f(u_{i+1})\big) = 1$ (consecutive positive numbers) for $7 \leq i \leq 11$,

$\gcd\big(f(v_i), f(v_{i+1})\big) = 1$ (consecutive positive numbers) for $7 \leq i \leq 11$,

$\gcd\big(f(u_i), f(v_i)\big) = 1$ for $7 \leq i \leq 12$ (k – prime labeling theorem)

$\gcd\big(f(u_{12}), f(v_{12})\big) = 1$ (consecutive positive numbers)

Since, $\gcd\big(f(u_{11}), f(v_{11})\big) = 1$ edge joining these two vertices can be removed.

Therefore, crab's right hand has a k – prime labeling.

Now, label Crab's legs as ①,②,③,④,⑤ and ⑥. Since each legs have 4 parts, there are 5 pair of vertices and last pair of vertices are not joined. Then in each leg $n = 5$.

In leg ①,

$x = 25 = u_{13}$ and since $n = 5$,

Using consecutive cyclic k – prime labeling theorem,

$2(x + n) - 1 = 2(25 + 5) - 1 = 59$ (prime number)

Therefore, leg ① has a consecutive cyclic prime labeling which ends with $x + 2n - 1 = 25 + 10 - 1 = 34 = v_{13}$.

Then, $\gcd\big(f(u_i), f(u_{i+1})\big) = 1$ (consecutive positive numbers) for $13 \leq i \leq 16$,

$\gcd\big(f(v_i), f(v_{i+1})\big) = 1$ (consecutive positive numbers) for $13 \leq i \leq 16$,

$\gcd\big(f(u_i), f(v_i)\big) = 1$ for $13 \leq i \leq 16$ (k – prime labeling theorem)

Therefore, leg ① has a k – prime labeling.

In leg ②,

$x = 35 = u_{18}$ and $n = 5$

Consider, $2(x + n) - 1 = 2(35 + 5) - 1 = 79$ (prime number)

Therefore, leg ② has a consecutive cyclic k – prime labeling, starting labeling with $u_{18} = 35$ and ending with $v_{18} = x + 2n - 1 = 35 + 10 - 1 = 44$ according to the method in the theorem.

i.e. $\gcd\big(f(u_i), f(u_{i+1})\big) = 1$ (consecutive positive numbers) for $18 \leq i \leq 21$,

$\gcd\big(f(v_i), f(v_{i+1})\big) = 1$ (consecutive positive numbers) for $18 \leq i \leq 21$,

$\gcd\big(f(u_i), f(v_i)\big) = 1$ for $18 \leq i \leq 21$ (k – prime labeling theorem)

In leg ③,

Let $x = 45 = u_{23}$ and

If $n = 5$, then $2(x + n) - 1 = 2(45 + 5) - 1 = 99$ which is not a prime.

Therefore, consecutive cyclic k – prime labeling theorem cannot be applied for leg ③.

But, if $n = 4$ (i.e. considering 4 pair of vertices in ladder shape leg), then $2(45 + 4) - 1 = 97$ (prime number)

Therefore, above theorem can be used to label a part of the leg and other two vertices can be labeled using suitable integers.

Using theorem, label first 4 pair of vertices starting with $u_{23} = 45$ and ending with $v_{23} = x + 2n - 1 = 45 + 8 - 1 = 52$ and label other remaining two vertices as $u_{27} = 53$ and $v_{27} = 54$.

i.e. $\gcd\big(f(u_i), f(u_{i+1})\big) = 1$ (consecutive positive numbers) for $23 \leq i \leq 25$,

$\gcd\big(f(v_i), f(v_{i+1})\big) = 1$ (consecutive positive numbers) for $23 \leq i \leq 25$,

$\gcd\big(f(u_i), f(v_i)\big) = 1$ for $23 \leq i \leq 26$ (k – prime labeling theorem)

If $u_{27} = 53$,

$$\gcd\big(f(u_{26}), f(u_{27})\big) = \gcd(48,53) = 1$$

If $v_{27} = 54$,

$$\gcd\big(f(v_{26}), f(v_{27})\big) = \gcd(49,54) = 1$$

Therefore, leg ③ has a $k -$ prime labeling.

To show leg ④ has a $k -$ prime labeling, similar process as for leg ③ can be used.

Hence, leg ④ has a $k -$ prime labeling with $u_{38} = 75, v_{38} = 82, u_{42} = 83$and $v_{42} = 84$.

And also, leg ⑤ and ⑥ can be labeled following the same procedure as for leg ①.

Leg ⑤ has a k - prime labeling, starting with $u_{43} = x = 85$ and ending with $v_{43} = 94$.

Further, by using the consecutive cyclic k – prime labeling theorem, leg ⑥ has a k - prime labeling, starting with $u_{48} = x = 95$ and ending with $v_{48} = 104$.

Now consider the Crab's swimming legs,

In right hand side swimming leg, label the first vertices as $u_{28} = 55$ and there is no any consecutive cyclic prime labeling for this leg.

Hence, label this leg as a walking leg using the labeling procedure for walking leg ③.

Let $n = 2$, then $2(x + n) - 1 = 2(55 + 2) - 1 = 113$ (prime number)

Hence, consecutive cyclic k – prime labeling method can be used to label first two pair of vertices starting with $u_{28} = 55$ and ending with $v_{28} = x + 2n - 1 = 55 + 4 - 1 = 58$. Remaining vertices can be labeled as a cycle starting with $u_{30} = 59$to $v_{30} = 64$.

Here, $\gcd\big(f(u_i), f(u_{i+1})\big) = 1$for $30 \le i \le 31$,

$\gcd(f(v_i), f(v_{i+1})) = 1$for $30 \le i \le 31$,

$\gcd\big(f(u_{32}), f(v_{32})\big) = \gcd(61,62) = 1$ (consecutive positive integers)

Note that,

$$\gcd\big(f(u_{30}), f(v_{30})\big) = \gcd(59,64) = 1$$

Therefore, u_{30} and v_{30} vertices can be combined using an edge.

Also, $\gcd\big(f(u_{29}), f(u_{30})\big) = \gcd(56,59) = 1$,

And $\gcd\big(f(v_{29}), f(v_{30})\big) = \gcd(57,64) = 1$.

i.e. Right hand side Crab's leg has a k – prime labeling.

In the left hand side Crab's swimming leg,

Let $u_{33} = 65 = x$. Consider this swimming leg as a walking leg.

There are 5 pair of vertices and $n = 5$.

Consider, $2(x + n) - 1 = 2(65 + 5) - 1 = 139$ (prime number)

Therefore, consecutive cyclic k – prime labeling method can be used.

Label left side swimming leg starting with $u_{33} = 65$ and ending with $v_{33} = x + 2n - 1 = 65 + 10 - 1 = 74$.

Since this is a swimming leg, edge between u_{36} and v_{36} should be removed.

Therefore, left side Crab's swimming leg also has a k – prime labeling.

Note that,

$\gcd\big(f(u_2), f(u_7)\big) = \gcd(2,13) = 1$,

$\gcd\big(f(v_2), f(v_{53})\big) = \gcd(11,116) = 1$,

$\gcd\big(f(v_7), f(u_{13})\big) = \gcd(24,25) = 1$,

$\gcd\big(f(v_{13}), f(u_{18})\big) = \gcd(34,35) = 1$,

$\gcd\big(f(v_{18}), f(u_{23})\big) = \gcd(44,45) = 1$,

$\gcd\big(f(v_{23}), f(u_{28})\big) = \gcd(52,55) = 1$,

$\gcd\big(f(v_{28}), f(u_{33})\big) = \gcd(58,65) = 1$,

$\gcd\big(f(v_{33}), f(u_{38})\big) = \gcd(74,75) = 1,$

$\gcd\big(f(v_{38}), f(u_{43})\big) = \gcd(82,85) = 1,$

$\gcd\big(f(v_{43}), f(u_{48})\big) = \gcd(94,95) = 1,$

$\gcd\big(f(v_{48}), f(u_{53})\big) = \gcd(104,105) = 1,$

$\gcd\big(f(v_{53}), f(v_{2})\big) = \gcd(116,11) = 1,$

Since every vertex joined by an edge in the above crab graph is relatively prime, it has a prime labeling.

CONCLUSION

In our work, a consecutive cyclic k – prime labeling for the ladder graph is obtained. Using the proved theorem any ladder graph can be labeled starting with any positive integer x satisfying the given conditions. Furthermore, we have introduced a new graph named as the Crab graph. A prime labeling have been discussed for this graph using the method proved for k – prime labeling of the ladder graph.

REFERENCES

[1] S. T. Arockiamary, G. Vijayalakshmi, k-Prime labeling of certain cycle connected graphs,| Malaya Journal of Matematik, vol. S, no. 1, pp. 280-283, 2019.

[2] A. H. Berliner, N. Dean, J. Hook, A. Marr, A. Mbirika, and C. D. McBee, Coprime and prime labeling of graphs,| Journal of Integer Sequences, vol. 19, no. 2, 2016.

[3] J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics,| vol. 16, no. 6, pp. 1-219, 2009.

[4] R. Tout, A. N. Dabbouchy, K. Howalla, Prime labeling of graphs,| national Academy of Science Letters-India, vol. 5, no. 11, pp. 365-368, 1982.

[5] T.R.D.S.M. Thennakoon, M.D.M.C.P. Weerarathna and A.A.I. Perera, Prime Labeling of NewlyConstructed Graph Using Star Graphs and Complete Bipartite Graphs,|Sumerianz Journal ofScientific Research, vol. 3, no. 2, pp. 10-17, 2020.

[6] S. K. Vaidya, and U. M. Prajapati, Some Results on Prime and k-Prime Labeling,| Journal of Mathematics Research, vol. 3, no. 1, February 2011.