Analytical Solutions of Optimal Control Problems with Mixed Constraints

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Abstract- In this research, the analytical solutions of optimal control problems constrained by ordinary differential equations are examined. The analytical solutions are obtained by applying the first order optimality conditions on the Hamiltonian function and solving the resulting system of first order ordinary differential equations. This leads to the optimal state, control and adjoint variables and hence the optimal objective function value. Two examples of optimal control problems constrained by ordinary differential equations are considered.

I. INTRODUCTION

Optimal control problems are mathematical programming problems involving state and control variables. Optimal control theory is used to minimize an objective function by finding a control law for a dynamical system over a given period of time. It has numerous applications in both science, engineering, epidemiology etc. Optimal control is a mathematical optimization method for deriving control policies [16]. It is an extension of the calculus of variations. The method is mainly due to the work of Lev Pontryagin and Richard Bellman in the 1950s.

In general, a quadratic constrained dynamic continuous optimization problem is defined as:

Minimize $J\big(x(t),u(t)\big) = \int_{t_0}^{t_f} f(t,x(t),u(t))dt$

Subject to $\dot{x}(t) = g\big(t,x(t),u(t)\big)$ (1.1)

$$x_{t_0} = x_0, t_0 \le t \le t_f$$

where $f\colon \Re \times \Re^n \times \Re^m \to \Re$; $g : \Re \times \Re^n \times \Re^m \to \Re^p$ are given functions and $m \le n$ and $p \le n$. The components of g are denoted by $g_1, g_2, \dots, g_p$ respectively.

Mathematical control theory deals with the basic principles underlying the analysis and design of control systems stemming from some areas of applied Mathematics. To control any system means to influence its behaviour so as to achieve a desired goal. Mathematical techniques are built to implement this influence [3].

Pontryagin's maximum (or minimum) principle, also known as the necessary condition, is a condition that must be satisfied for a statement to be established. However, the condition does not validate the statement. For the validity of the statement to be established, the sufficient conditions must be satisfied in order to determine the nature of the turning point ([6]). Pontryagin's maximum (or minimum) principle was formulated in 1956 by the Russian mathematician Lev Pontryagin and his students as Pontryagin's maximum principle and proved historically based on maximizing the Hamiltonian [4].

Optimal Control involves determining control and state trajectories for a dynamic system over a period of time in order to minimize an objective function, cost functional or a performance index. The theory of optimal control is well developed for many decades ([18]). It requires a performance index or cost functional $J\big(x(t),u(t)\big)$, a set of state variables $(t, x(t) \in X)$, a set of control variables $(t, u(t) \in U)$ in a time t, with $t_0 \le t \le t_f$. To maximized a given objective functional, one must obtain the associated state variable $x(t)$ and a piecewise continuous control $x(t)$.

Many optimization techniques have been developed over the years for the solutions of optimization problems. In the early years of optimal control, the favoured approach for solving optimal control problems was that of indirect method. In an indirect method, the calculus of variations is employed to obtain the first-order optimality conditions ([9]).

Several well-defined new areas in optimization theory were reviewed in [12]. A number of numerical techniques have been developed over the years for optimal control problems that have analytical solutions and the ones that did not have analytical solutions by [1], [2], [5], [11], [13], [14], [15] and [16].

[7] considered an optimal control problem involving equality and inequality state-control constraints with fixed initial and final endpoint state constraints. The control variables belong to the class of piecewise continuous functions while and state variables belong to the class of piecewise smooth functions respectively. The paper provided a direct derivation of second order necessary conditions, based on a variational approach where a certain quadratic form is non-negative. The inequality constraints are treated as equalities of the active constraints for this set, thus producing a restrictive set of optimality conditions.

[10] obtained the analytic and numeric solutions of discretized constrained optimal control problems. The analytic and numeric solutions of general continuous linear quadratic optimal control problem were presented. The associated general Riccati differential equation was solved by numerical-analytical approach using variational iteration method. Numerical solutions of the constrained optimal control problem were obtained via quadratic programming of the discretized continuous optimal control problems by by shooting method and the conjugate gradient method (CGM). The results showed that both the analytical and numerical solutions agreed favourably. In the paper, the analytical solution and numerical solution by penalty function methods were presented and convergence analysis was conducted.

II. MATERIALS AND METHODS

The general form of optimal control problem constrained by n equality constraints is given as

Minimize $I(x,u) = \int_0^T f(x,u,t)dt$

Subject to $\dot{x}_i(t) = h_i(x,u,t),\ i = 1,2,...,n$ (2.1)

where $x \in \mathfrak{R}^n,\ u \in \mathfrak{R}^m$, f: $\mathfrak{R} \times \mathfrak{R}^n \times \mathfrak{R}^m \to \mathfrak{R}$ and g : $\mathfrak{R} \times \mathfrak{R}^n \times \mathfrak{R}^m \to \mathfrak{R}^p$.

2.1 Necessary Conditions for General Optimal Control Problem with Equality Constraints.

The standard form of a general optimal control problem constrained by n equality constraints is given as:

Minimize $I(x,u) = \int_0^T f(x,u,t)dt$

Subject to $\dot{x}_i(t) = h_i(x,u,t),\ i = 1,2,...,n$ (2.2)

where $x \in \mathfrak{R}^n$ and $u \in \mathfrak{R}^m$.

$$I^*(x,u) = \int_0^T \Big(f(x,u,t) + \sum_{i=1}^n \lambda_i\big(h_i(t,x,u) - \dot{x}_i(t)\big)\Big)dt \quad (2.3)$$

The necessary conditions required to solve (2.2) is derived from the Hamiltonian which is given as

$$H(t,x,u,\lambda) = f(t,x,u) + \sum_{i=1}^n \lambda_i h_i(t,x,u) \quad (2.3)$$

where λ_i are the adjoint variables and are dependent on t, x and u.

The Euler-Lagrange equations for (2.4) are given as

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}_i}\right) = 0,\ i = 1,2,\ldots,n \quad (2.5)$$

$$\frac{\partial H}{\partial u_j} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{u}_j}\right) = 0,\ j = 1,2,\ldots,m \quad (2.6)$$

$$\frac{\partial H}{\partial \lambda_i} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{\lambda}_i}\right) = 0,\ i = 1,2,\ldots,n \quad (2.7)$$

Applying equations (2.5), (2.6) and (2.7) on (2.4), we have

$$\frac{\partial f}{\partial x_i} + \lambda_i \frac{\partial h_i}{\partial x_i} + \dot{\lambda}_i = 0, \quad i = 1,2,\ldots,n \quad (2.8)$$

$$\frac{\partial f}{\partial u_j} + \lambda_i \frac{\partial h_i}{\partial u_j} = 0, \quad j = 1,2,\ldots,m \quad (2.9)$$

$$h_i(x,u,t) - \dot{x}_i(t) = 0, \quad i = 1,2,\ldots,n \quad (2.10)$$

Hence,

$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$ Adjoint Equation (2.11)

$\frac{\partial H}{\partial x_i} = 0$ Optimality Condition (2.12)

$\dot{x}(t) = h_i(x,u,t)$ State Equation (2.13)

Equations (2.11) and (2.13) are system of first order ordinary differential equations that can be solved simultaneously in order to get the values of the two constants by applying the boundary conditions. If only one boundary condition is given, then the free end condition is applied to get the second boundary condition. The free-end condition is given as

$\frac{\partial H}{\partial x_i} = 0$ or $\lambda_i(T) = 0$ Transversality Condition (2.14)

For the solution to be optimal, all the conditions given by equations (2.11), (2.12), (2.13) and (2.14) must be satisfied. If one or more of the conditions are not satisfied, then the solution is not optimal.

2.2 Analytical Solution of Optimal Control Problems Constrained by two Equality Ordinary Differential Equations

Minimize $I(x_1, x_2, u) = \int_0^T (px_1^2(t) + qx_2^2(t) + wu^2(t))dt$ (2.15)

Subject to $\dot{x_1}(t) = ax_1(t) + bx_2(t) + cu(t)$ (2.16)

$\dot{x}_2(t) = dx_1(t) + ex_2(t) + fu(t)$ (2.17)

$x_1(0) = x_{10}, x_2(0) = x_{20}, 0 \leq t \leq T$

where a, b, c, d, e, f are real constants and $p, q, w > 0$.

Theorem 0.2.1. Given the optimal $u^*(t)$ and the solutions x_1^* and x_2^* of the state system (2.16) and (2.17) that minimizes $I(x_1, x_2, u)$ over U (where U is the permissible set of controls), then there exists adjoint variables $\mu_1(t)$ and $\mu_2(t)$ satisfying

$\dot{\mu}_1(t) = -2px_1(t) - a\mu_1(t) - d\mu_2(t), t \in [0, T]$ (2.18)

$\dot{\mu}_2(t) = -2qx_2(t) - b\mu_1(t) - e\mu_2(t), t \in [0, T]$ (2.19)

and with the transversality conditions

$\mu_1(T) = 0, \mu_2(T) = 0$ (2.20)

$u^*(t) = \frac{-c\mu_1(t) - f\mu_2(t)}{2w}$ (2.21)

Proof. With appropriate conditions on the end points, adjoint variables $\mu_1(t)$ and $\mu_2(t)$ can be introduced by forming the required augumented functional from (2.15)-(2.17).

The Hamiltonian function is given as

$H(x_1, x_2, u, \mu_1, \mu_2) = px_1^2(t) + qx_2^2(t) + wu^2(t) + \mu_1(ax_1(t) + bx_2(t) + cu(t)) + \mu_2(dx_1(t) + ex_2(t) + fu(t))$ (2.22)

From the knowledge of calculus of variation, it seems plausible that the optimal solutions for our initial optimal control problem ought to be the Euler-Lagrangian equations for H regarded as function of five variables $(x_1, x_2, u, \mu_1, \mu_2)$. Thus, the E-L system can be written as

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{x_1}}] = \frac{\partial H}{\partial x_1} \quad (2.23)$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{x_2}}] = \frac{\partial H}{\partial x_2} \quad (2.24)$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{u}}] = \frac{\partial H}{\partial u} \quad (2.25)$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{\mu_1}}] = \frac{\partial H}{\partial \mu_1} \quad (2.26)$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{\mu_2}}] = \frac{\partial H}{\partial u_2} \quad (2.27)$$

Equations (2.23)-(2.27) together with (2.21) now give the following system of first order ordinary differential equations:

$\dot{x}_1(t) = ax_1(t) + bx_2(t) - c(\frac{c\mu_1(t) + f\mu_2(t)}{2w})$ (2.28)

$\dot{x}_2(t) = dx_1(t) + ex_2(t) - f(\frac{c\mu_1(t) + f\mu_2(t)}{2w})$ (2.29)

$\dot{\mu}_1(t) = -2px_1(t) - a\mu_1(t) - d\mu_2(t)$ (2.30)

$\dot{\mu}_2(t) = -2qx_2(t) - b\mu_1(t) - e\mu_2(t)$ (2.31)

The optimality system of equations (2.28)-(2.31) represent a linear 4-point boundary value differential equations, which are the necessary conditions for an optimal control $u^*(t)$. The general solution is given by

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{\mu_1}(t) \\ \dot{\mu_2}(t) \end{pmatrix} = e^{tM} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \mu_1(0) \\ \mu_2(0) \end{pmatrix} \quad (2.32)$$

where

$$M = \begin{pmatrix} a & b & \frac{-c^2}{2w} & \frac{-cf}{2w} \\ d & e & \frac{-cf}{2w} & \frac{-f^2}{2w} \\ -2p & 0 & -a & -d \\ 0 & -2q & -b & -e \end{pmatrix} \quad (2.33)$$

Since we know $x_1(0)$ and $x_2(0)$, the task is to choose $\mu_1(T)$ and $\mu_2(T)$ so that the transversality condition is satisfied. The characteristics equation of M is given by

$$\lambda^4 - \left(\frac{c^2p + f^2q + w(a^2 + e^2 + 2bd)}{w}\right)\lambda^2 + \left(w\left(a^2e^2 + bd(bd - 2ae)\right)\right) + \frac{f^2(a^2q+b^2p)+c^2(d^2q+e^2p)-2cf(adq+bep)}{w} = 0 \quad (2.34)$$

and its eigenvalues are

$$\lambda_1 = -\sqrt{\frac{\left(-\sqrt{(A+B+C)} + c^2p + f^2q + a^2w + e^2w + 2bdw\right)}{2w}} \quad (2.35)$$

$$\lambda_2 = \sqrt{\frac{\sqrt{(A+B+C)} + c^2p + f^2q + a^2w + e^2w + 2bdw}{2w}} \quad (2.36)$$

$$\lambda_3 = -\sqrt{\frac{\left(\sqrt{(A+B+C)} + c^2p + f^2q + a^2w + e^2w + 2bdw\right)}{2w}} \quad (2.37)$$

$$\lambda_4 = \sqrt{\frac{-\sqrt{(A+B+C)} + c^2p + f^2q + a^2w + e^2w + 2bdw}{2w}} \quad (2.38)$$

where

$A = a^4w^2 + 4a^2bdw^2 + 2a^2c^2pw - 2a^2e^2w^2 - 2a^2f^2qw + 8abdew^2 + 8acdfqw$ (2.39)

$B = 4bc^2dpw - 4b^2f^2pw + 8bcefpw + 4bde^2w^2 + 4bdf^2qw + c^4p^2 - 4c^2d^2qw$ (2.40)

$C = 2c^2f^2pq - 2c^2e^2pw + e^4w^2 + 2e^2f^2qw + f^4q^2$ (2.41)

These eigenvalues are used to obtain the eigenvectors $\overrightarrow{U_1}$, $\overrightarrow{U_2}$, $\overrightarrow{U_3}$ and $\overrightarrow{U_4}$. Thus, the general solution of equation (2.32) is

$$V(t) = c_1\overrightarrow{U_1}e^{\lambda_1 t} + c_2\overrightarrow{U_2}e^{\lambda_2 t} + c_3\overrightarrow{U_3}e^{\lambda_3 t} + c_4\overrightarrow{U_4}e^{\lambda} \quad (2.42)$$

where c_1, c_2, c_3 and c_4 are the constants of integration. Thus, c_1, c_2, c_3 and c_4 can be determined by substituting the initial and terminal conditions into equation (2.42).

III. RESULTS

Example 1.

Minimize $I(x_1, x_2, u) = \int_0^1 (x_1^2(t) + x_2^2(t) + 0.005u^2(t))dt$ (3.1)

subject to $\dot{x}_1(t) = x_2(t)$ (3.2)

$\dot{x}_2(t) = -x_2(t) + u(t)$ (3.3)

with initial conditions $x_1(0) = 0, x_2(0) = -1$ (3.4)

Solution 1. The Hamiltonian function, H, is given by

$$H(x_1, x_2, u, \mu_1, \mu_2) = x_1^2(t) + x_2^2(t) + 0.005u^2(t) + \mu_1 x_2(t) + \mu_2(-x_2(t) + u(t)) \quad (3.5)$$

From the knowledge of calculus of variations, it seems plausible that the optimal solutions for our initial optimal control problem ought to be the Euler-Lagrange equations for H regarded as a function of five variables $(x_1, x_2, u, \mu_1, \mu_2)$. Thus, the E-L system can be written as

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x_1}}\right] = \frac{\partial H}{\partial x_1} \quad (3.6)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x_2}}\right] = \frac{\partial H}{\partial x_2} \quad (3.7)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{u}}\right] = \frac{\partial H}{\partial u} \quad (3.8)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{\mu_1}}\right] = \frac{\partial H}{\partial \mu_1} \quad (3.9)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{\mu_2}}\right] = \frac{\partial H}{\partial u_2} \quad (3.10)$$

Equations (3.6)-(3.10) together with (3.2) and (3.3) now give the following system of first order ordinary differential equations;

$\dot{x}_1(t) = x_2(t)$ (3.11)

$\dot{x}_2(t) = -x_2(t) - 100\mu_2(t)$ (3.12)

$u(t) = -100\mu_2(t)$ (3.13)

$\dot{\mu}_1(t) = 2x_1(t)$ (3.14)

$\dot{\mu}_2(t) = -2x_2(t) - \mu_1(t) + \mu_2(t)$ (3.15)

The optimality system of equations (3.11)-(3.12) and (3.14)-(3.15) represent a linear 4-point boundary value differential equations, which are the necessary conditions for an optimal control $u^*(t)$ with an algebraic equation (3.13). These equations can be written in matrix form as

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{\mu_1}(t) \\ \dot{\mu_2}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -100 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} \quad (3.16)$$

The general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \mu_1(t) \\ \mu_2(t) \end{pmatrix} = e^{tM} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \mu_1(0) \\ \mu_2(0) \end{pmatrix} \qquad (3.17)$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -100 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 \end{pmatrix} \qquad (3.18)$$

Since we know $x_1(0)$ and $x_2(0)$, the task is to choose $\mu_1(1)$ and $\mu_2(1)$ using the free-end condition so that the transversality condition is satisfied. The characteristics equation of M is given by $\lambda^4 - 201\lambda^2 + 200 = 0$ and its eigenvalues are $\lambda_1 = \frac{2786}{197}$, $\lambda_2 = 1$, $\lambda_3 = -1$ and $\lambda_4 = -\frac{2786}{197}$. The corresponding eigenvectors are obtained and substituted into the general solution to obtain

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \mu_1(t) \\ \mu_2(t) \end{pmatrix} = c_1 e^{\frac{2786}{197}t} \begin{pmatrix} \frac{652}{9349} \\ \frac{503}{510} \\ \frac{-97}{9835} \\ \frac{-125}{837} \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \frac{-932}{2283} \\ \frac{-932}{2283} \\ \frac{1517}{1858} \\ \frac{140}{17147} \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \frac{881}{2158} \\ \frac{-881}{2158} \\ \frac{881}{1079} \\ 0 \end{pmatrix} + c_4 e^{-\frac{2786}{197}t} \begin{pmatrix} \frac{-187}{2674} \\ \frac{899}{909} \\ \frac{-89}{8999} \\ \frac{160}{1231} \end{pmatrix} \qquad (3.19)$$

where c_1, c_2, c_3 and c_4 are constants of integration. Thus, c_1, c_2, c_3 and c_4 can be determined by substituting the initial and terminal conditions $x_1(0) = 0$ and $x_2(0) = -1$, $\mu_1(1) = 0$ and $\mu_2(1) = 0$ into equation (3.19). Hence, $c_1 = \frac{2}{854612229}$, $c_2 = \frac{9692}{443877}$, $c_3 = -\frac{3379}{20958}$ and $c_4 = -\frac{62831}{58794}$.

Therefore, the particular solution becomes

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \mu_1(t) \\ \mu_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1304}{7989769728921} \\ \frac{503}{217926118395} \\ \frac{-194}{8405111272215} \\ \frac{-250}{715310435673} \end{pmatrix} e^{\frac{2786}{197}t} + \begin{pmatrix} \frac{-9032944}{1013371191} \\ \frac{-9032944}{1013371191} \\ \frac{7351382}{412361733} \\ \frac{193840}{1087308417} \end{pmatrix} e^{t} + \begin{pmatrix} \frac{-2976899}{45227364} \\ \frac{2976899}{45227364} \\ \frac{-2976899}{22613682} \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} \frac{11749397}{157215156} \\ \frac{-56485069}{53443746} \\ \frac{5591959}{529087206} \\ \frac{-5026480}{36187707} \end{pmatrix} e^{-\frac{2786}{197}t} \qquad (3.20)$$

The analytical objective function value is 0.1732.

Example 2

Minimize $\quad I(x_1, x_2, u) = \int_0^1 (0.5x_1^2(t) + 2x_2^2(t) + u^2(t))dt \qquad (3.21)$

Subject to $\dot{x}_1(t) = x_1 + 2u(t) \qquad (3.22)$

$\dot{x}_2(t) = x_2(t) \qquad (3.23)$

with initial conditions $x_1(0) = 1, x_2(0) = 1$ (3.24)

Solution 2. The Hamiltonian function function, H, is given by

$$H(x_1, x_2, u, \mu_1, \mu_2) = 0.5x_1^2(t) + 2x_2^2(t) + u^2(t) + \mu_1(x_1(t) + 2u(t)) + \mu_2(x_2(t)) \qquad (3.25)$$

From the knowledge of calculus of variation, it seems plausible that the optimal solutions for our initial optimal control problem ought to be the Euler-Lagrange equations for H regarded as a function of five variables $(x_1, x_2, u, \mu_1, \mu_2)$. Thus, the E-L system can be written as

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x_1}}\right] = \frac{\partial H}{\partial x_1} \quad (3.26)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x_2}}\right] = \frac{\partial H}{\partial x_2} \quad (3.27)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{u}}\right] = \frac{\partial H}{\partial u} \quad (3.28)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{\mu_1}}\right] = \frac{\partial H}{\partial \mu_1} \quad (3.29)$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{\mu_2}}\right] = \frac{\partial H}{\partial u_2} \quad (3.30)$$

Equations (3.26)-(3.30) together with (3.2) and (3.3) now give the following system of first order ordinary differential equations

$$\dot{x}_1(t) = x_1(t) - 2\mu_1(t) \quad (3.31)$$

$$\dot{x}_2(t) = x_2(t) \quad (3.32)$$

$$u(t) = -\mu_2 \quad (3.33)$$

$$\dot{\mu}_1(t) = -x_1(t) - \mu_1(t) \quad (3.34)$$

$$\dot{\mu}_2(t) = -4x_2(t) - 2\mu_2(t) \quad (3.35)$$

The optimality system of equations (3.31)-(3.32) and (3.34)-(3.35) represent a linear 4-point boundary value differential algebraic equations, which are the necessary conditions for an optimal control u*(t). These equations can be written in matrix form as

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{\mu_1}(t) \\ \dot{\mu_2}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -4 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \mu_1 \\ \mu_2 \end{pmatrix}$$

(3.36)

The general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \mu_1(t) \\ \mu_2(t) \end{pmatrix} = e^{tM} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \mu_1(0) \\ \mu_2(0) \end{pmatrix} \quad (3.37)$$

where

$$M = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -4 & 0 & -2 \end{pmatrix} \quad (3.38)$$

Since we know $x_1(0)$ and $x_2(0)$, the task is to choose $\mu_1(1)$ and $\mu_2(1)$ using the free-end condition so that the transversality condition is satisfied. The characteristics equation of M is given by $\lambda^4 + \lambda^3 - 5\lambda^2 - 3\lambda + 6 = 0$ and its eigenvalues are $\lambda_1 = \frac{1351}{780}, \lambda_2 = -\frac{1351}{780}, \lambda_3 = -2$ and $\lambda_4 = 1$. The corresponding eigenvectors are obtained and substituted into the general solution to obtain

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \mu_1(t) \\ \mu_2(t) \end{pmatrix} = c_1 e^{\frac{1351}{780}t} \begin{pmatrix} \frac{1233}{1313} \\ 0 \\ -\frac{408}{1187} \\ 0 \end{pmatrix} + c_2 e^{-\frac{1351}{780}t} \begin{pmatrix} \frac{368}{623} \\ 0 \\ \frac{3322}{4117} \\ 0 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{t} \begin{pmatrix} 0 \\ \frac{3}{5} \\ 0 \\ -\frac{4}{5} \end{pmatrix} \quad (3.39)$$

where c_1, c_2, c_3 and c_4 are constants of integration. Thus, c_1, c_2, c_3 and c_4 can be determined by substituting the initial and terminal conditions $x_1(0) = 1$ and $x_2(0) = 1$, $\mu_1(1) = 0$ and $\mu_2(1) = 0$ into equation (3.39). Hence, $c_1 = \frac{3931}{35292}$, $c_2 = \frac{66202}{43673}$, $c_3 = \frac{83047}{3101}$ and $c_4 = \frac{5}{3}$.

Therefore, the particular solution becomes

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \mu_1(t) \\ \mu_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1615641}{15446132} \\ 0 \\ -\frac{7862}{205351} \\ 0 \end{pmatrix} e^{\frac{1351}{780}t} + \begin{pmatrix} \frac{24362336}{27208279} \\ 0 \\ \frac{219923044}{179801741} \\ 0 \end{pmatrix} e^{-\frac{1351}{780}t} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{83047}{3101} \end{pmatrix} e^{-2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{4}{3} \end{pmatrix} e^{t} \quad (3.40)$$

The analytical solution is 0.5745.

CONCLUSION

The analytical solutions of continuous quadratic optimal control problems constrained by ordinary differential equations have been presented. The proposed analytical method gives the exact solution of this class of optimal control problems. It is therefore recommended that the method will be extended to solve real life problems.

REFERENCES

[1] Batiha B., Noorani M. S. M. and Hashim I. (2007). *Application of Variational Iteration Method to a General Riccati Equation*. International Mathematical Forum. Volume 2, Pages 2759-2770.

[2] Castelli, R., Lessard J. and James J. D. M. (2015), *Analytic Enclosure of the Fundamental Matrix Solution*. Conference Applications of Mathematics 2015, in honor of the birthday anniversaries of Ivo Babuska (90), Milan Prager (85), and Emil Vitasek (85) J. Brandts, S. Korotov, M. Krzek, K. Segeth, J. Sstek, T. Vejchodsky (Eds.) , Institute of Mathematics AS CR, Prague 2015, Pages 1-19.

[3] Claudiu, C. R. (2006), *Linear Control*. Rhodes University, Grahamstown 6140, South Africa Pages 1-217.

[4] Dmitruk, Andrei V. (2009), *On the development of Pontryagin's Maximum Principle in the works of A.Ya. Dubovitskii and AA Milyutin*. Control and cybernetic, Volume 38, Number 4A. Pages 1-38.

[5] Fair, R.C. (1973), *On the Solution of Optimal Control Problems as Maximization Problems*. Econometric Research Program, Research Memorandum, Number 146, Pages 1-32.

[6] Ioffe Alexander D. (2020), *An Elementary Proof of the Pontryagin's Maximum Principle*. Vietnam Journal of Mathematics, 1-10.

[7] Javier, F.R. (2009), *Equality-Inequality Mixed Constraints in Optimal Control*. International Journal of Mathematical Analysis, Volume 3, Number 28, Pages 1369-1387.

[8] Mattheij, R. and Molenaar J. (2002), *Ordinary Differential Equations in Theory and Practice*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia Pages 231-237.

[9] Naidu, D.S. (2003), *Optimal Control Systems*. CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida, Pages 1-96.

[10] Olotu, O. and Adekunle, A. I. (2010), *Analytic and Numeric Solutions of Discretized Constrained Optimal Control Problem with Vector-Matrix Coefficients*.
Mathematics Subject Classification, AMO-Advanced Modeling and Optimization, Volume 12, Number 1, Pages 119-131.

[11] Olotu, O. and Afolabi, A.S. (2014), *A New Scheme Using Penalty Functions with B-Spline Function for Constrained Optimal Control Problems*. International Journal of Numerical Mathematics (IJNM), Volume 7, Number 2, 282-300.

[12] Polak, E. (1973), *An Historical Survey of Computational Methods in Optimal Control*. SIAM Review, Volume 15, Number 2, Pages 552-584.

[13] Polak, E. (1971), *Computational Methods in Optimization: A Unified Approach*. Academic Press, New York, Pages 1-50.

[14] Radhi, A.Z., Shiv, P.Y and Mohan, C. (1999), *Penalty Method for an Optimal Control Problem with Equality and Inequality Constraints*. Indian Journal of Pure and Applied Mathematics, Volume 30, Number 1, Pages 1-14.

[15] Serrezuela, R. R. (2016), *The K-Exponential Matrix to solve systems of differential equations deformed*. Global Journal of Pure and Applied Mathematics, Volume 12, Number 3, Pages 1921-1945. Singiresu, S. R.(1996), *Engineering Optimization: Theory and Practice*. Third Edition, West Lafayette, Indiana, Pages 1-450.

[16] Suha, N. A., Fuad, A. A. and Saba, S. H. (2010), *A New Computational Method for Optimal Control Problem with B-spline Polynomials*. Eng. and Tech Journal, Volume 28, Number 2, Pages 5711-5718.

[17] Vincent, H. (2005), *Optimal Control*. Retrieved from
https://en.wikipedia.org/wiki/Optimal_control
on 13/11/2016 at 22:02 GMT.