Some Linear Codes and Designs from Maximal Subgroup of Degree 1288 Related to Mathieu Group M_{24}

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Abstract- In this paper, we have recursively found all submodules of the permutation module of degree 1288. The permutation module split into 252 submodules. We constructed the partial lattice diagram of the submodules. We discussed four non trivial submodules of dimensions 11, 12, 22 and 23. We determined designs of minimum weight in $C_{1288,\,i}$. We examined all symmetric 1-designs invariant under Mathieu group M_{24}.

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Indexed Terms- Designs, Linear codes, Mathieu Group M_{24}, Maximal sub-group.

I. INTRODUCTION

From the action of the permutation group G on a finite set Ω, of degree 1288, we construct a 1288-dimensional permutation module invariant under *G*. We take the permutation module to be our working module and recursively find all submodules. The permutation module splits into 252 submodules. These submodules represent the dimensions of the codes associated with a module of length 1288. The module breaks into nine completely irreducible parts of size 1, 11,11, 44, 44, 120, 220, 220 and 252 of multiplicities 4, 4, 4, 3, 3, 2, 1, 1 and 1 respectively. Table 1 shows the 1st 140 submodules with the dimension k from this permutation module.

Table 1: The 1st 140 Submodules with the Dimension k from 1288 Permutation Module.

k	#	k	#	k	#	k	#	k	#	k	#
0	1	112	1	276	2	407	1	474	1	539	3
1	1	186	1	277	1	417	1	483	3	540	3
11	1	187	1	286	2	418	1	484	4	551	1
12	1	230	1	287	3	428	1	485	1	561	1
22	1	231	4	288	1	429	1	494	1	562	1
23	1	232	2	297	2	450	1	495	2	572	1
45	1	241	1	298	3	451	3	496	1	573	2
56	1	242	2	299	1	452	1	506	3	574	1
57	1	243	1	331	1	461	3	507	3	583	1
66	1	252	1	332	1	462	5	517	5	584	2
67	2	253	2	342	1	463	1	518	4	585	1
68	1	254	1	343	1	472	4	528	5	595	1
111	1	275	1	406	1	473	6	529	5	596	1

The remaining submodules are of dimension $n - k$
Partial submodule lattice is as shown in Figure 1

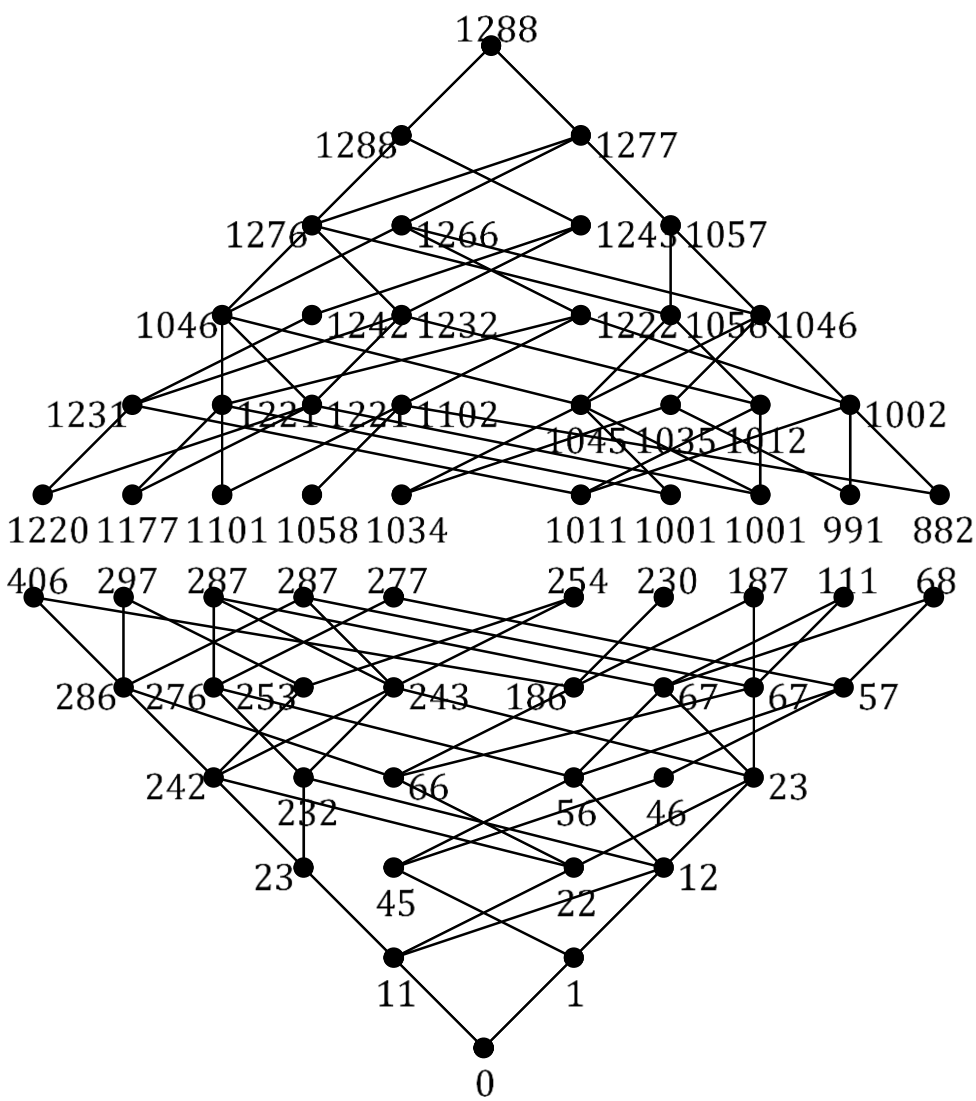

Figure 1: Submodule Lattice of degree 1288

From the lattice structure, submodules of dimension 1 and dimension 11 are irreducible.

II. SOME BINARY LINEAR CODES

We discuss four non trivial submodules of small dimensions 11, 12, 22, and 23. The binary linear codes of these submodules are represented in table 2.

Table 2: Codes of small dimensions

Name	Dimension	Parameters
C1288,1	11	$[1288,11,640]_2$
C1288,2	12	$[1288,12,616]_2$
C1288,3	22	$[1288,22,448]_2$
C1288,4	23	$[1288,23,448]_2$

We make some observations about the codes. These properties are examined with certain detail in Proposition 2.1.

- Proposition 2.1. Let G be the mathieu group M_{24} and $C_{1288,1}$, $C_{1288,2}$, $C_{1288,3}$ be nontrivial binary codes of dimension 11, 12, 22 respectively derived from a module of length 1288. Then the following holds;

i. $C_{1288,1}$ is a doubly even, self-orthogonal and projective [1288, 11,640] binary code. The dual code of $C_{1288,1}$ is [1288, 1277, 3] binary code.

ii. $C_{1288,2}$ is a doubly even, self-orthogonal and projective [1288, 12,616] binary code. The dual code of $C_{1288,2}$ is [1288, 1276, 4] binary code.

iii. $C_{1288,3}$ is a doubly even, self-orthogonal and projective [1288, 23, 448] binary code. The dual code of $C_{1288,3}$ is [1288, 1265, 4] binary.

iv. $C_{1288,4}$ is doubly even, self-orthogonal projective [1288, 23, 448] binary code. The dual code of C1288,4 is a [1288, 1265, 4].

- Proof

i. The polynomial of this code $C_{1288,1}$ is $W_{C1288,1} = 1+1771x^{640}+276x^{672}$. We observe that both codewords have weights divisible by 4. Hence $C_{1288,1}$ is self-orthogonal. The minimum weight of $C^{\perp}_{1288,1}$ code is 3. Hence $C_{1288,1}$ is projective.

ii. The weight distribution of this code $C_{1288,2}$ is $W_{C1288,2} = 1 + 276x^{616} + 1771x^{648} +276x^{672}+x^{1288}$. We observe that both codewords have weights divisible by 4. Hence $C_{1288,2}$ is self-orthogonal. The minimum weight of $C^{\perp}_{1288,2}$ code is 4. Hence $C_{1288,2}$ is projective.

iii. The polynomial of this code $C_{1288,3}$ is $WC1288,3$ = 1+759x448+26565x576+170016x600+97152x616 +510048x632+1772771x640+ $680064x^{648}+637560x^{664}+276828x^{672}+21252x^{736}+1288x^{792}$. Since the weight of all codewords of $C_{1288,3}$ is divisible by 4, $C_{1288,3}$ is doubly even. Hence $C_{1288,3}$ is self-orthogonal. The minimum weight of $C^{\perp}_{1288,3}$ code is 4. Hence $C_{1288,3}$ is projective.

iv. The polynomial of this code $C_{1288,4}$ is $WC1288,3$ = 1+759x448+1288x496+21252x552+26565x576+170016x600+373980x616+ $637560x^{624} + 510048x^{632} + 2452835x^{640} + 2452835x^{648} + 510048x^{656} + 637560x^{664} + 373980x^{672} +170016x^{688} +26565x^{712} +21252x^{736} +1288x^{792} +759x^{840} +x^{1288}$. Since weights of all codewords of $C_{1288,4}$ is divisible by 4, $C_{1288,4}$ is doubly even. Hence $C_{1288,4}$ is self-orthogonal. The minimum weight of $C^{\perp}_{1288,4}$ code is 4. Hence $C_{1288,4}$ is projective. Since $C^{\perp}_{1288,1}$ is a two-weight code, a strongly regular graph Γ ($C^{\perp}_{1288,1}$) is obtained.

- Lemma 2.2. $T(C^{\perp}_{1288,1})$ is a strongly regular [2048, 1288, 792,840] graph with spectrum $[1288]^1,[8]^{1771}, [-56]^{276}$.
- Remark 2.3. This graph has not been mentioned by Calderbank.
- Remark 2.4. Codewords of $C_{1288,1}$ and codewords of $C^{\perp}_{1288,1}$ is discussed geometrically;

i. The words of $C_{1288,1}$ is the the blocks of the design D_{640}.

ii. The words of weight 640 in $C_{1288,1}$ is isomorphic to $2^{6:}$ 3.S_6.

iii. The codewords of weight 672 in $C_{1288,1}$ is isomorphic to $M_{12:}$ 2.

III. SOME DESIGNS OF MINIMUM WEIGHT

We determine designs of minimum weight in $C_{1288,\ i}$. Table 3 shows Deigns of minimum weight in $C_{1288,\ i}$. Column one indicates the code of weight m and column two shows the parameters of the 1-designs. In column three we give the number of blocks and column four we list the automorphism group of the design.

Table 3: Deigns of minimum weight in $C_{1288,\ i}$

Code	Dw_m	No of Blocks	Aut(D_{wm})
[1288, 11, 640]	1-(1288, 640, 880)	1771	M_{24}
[1288, 12, 616]	1-(1288, 616, 138)	276	M_{24}
[1288, 22, 448]	1-(1288, 448, 264)	759	M_{24}
[1288, 23, 448]	1-(1288, 448, 264)	759	M_{24}

Table 3 shows that t-designs of minimum weights are isomorphic to M_{24}

IV. SYMMETRIC 1-DESIGN

In this section we examine all symmetric 1-designs invariant under Mathieu group M_{24} as constructed from orbits of the stabilizer. Table 4 shows Designs from primitive group of degree 1288. Column one shows the 1-design D_k of orbit length k, column two represents the orbit length, the third column gives the parameters of the 1-designs D_k and column four gives the automorphism group of the design.

Table 4: Designs from primitive group of degree 1288

Design	orbit length	parameters	Automorphism Group
D495	495	1-(1288,495,495)	M_{24}
D792	792	1-(1288,792,792)	M_{24}
D496	496	1-(1288,496,496)	M_{24}
D793	793	1-(1288,793,793)	M_{24}
D1287	1287	1-(1288,1287,1287)	

- Proposition 4.1. Let G be the mathieu simple group M_{24}, and Ω the primitive G-set of size 1288 defined by the action on the costs of M_{12}:2. Let β = {M^g: g ∈ G} and D_k = (Ω, β). Define the sets M and N such that M = {495,792,496,793} and N = {1287}. If k ∈ M then Aut(D_k) ~= M_{24}
- Proof First, we consider the case when k ∈ M. The only composition factor of Aut (D_k) is M_{24}. This implies that Aut (D_k) ~= M_{24}

CONCLUSION

Let G be the primitive group of degree 1288 of M_{24} and C a linear code.

Then there exist:

a. a set of Primitive Designs related to M_{24}.
b. a set of primitive symmetric 1-designs
c. a strongly regular graphs related to two weight codes
d. a set of self-orthogonal doubly even projective codes

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