Prime Labeling of Circular Ladder Graph (CL_n)

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Abstract- ***Prime labeling is an assignment of integers to the vertices which are relatively prime. The theory behind the graph labeling were introduced by Rosa in the 1960s. A huge number of prime labeling research works have been discovered for various types of graphs today after undergoing different methods. In this work, the prime labeling method has been introduced for the circular ladder graph CL_n with $2n$ vertices when n is an even. The graph obtained by using cartesian product of C_n with n vertices and path graph of the form P_2 is called circular ladder graph and is denoted by CL_n (i.e $C_n \times P_2 = CL_n$). We proved that CL_n is a prime graph for two special cases when n is an even integer and $n+1$ is prime, and when n is an even integer and $2n+1$ is prime. In addition, the cyclic vertex labeling method in the clockwise direction and anti-clockwise direction have been introduced. Moreover, proofs of the two theorems related to the above two cases are given. We used the theorem on ladder graphs in the work by A. H. Berliner et al. to prove these results.***

Indexed Terms- ***Circular Ladder Graph, Graph Labeling, Prime Labeling***

I. INTRODUCTION

Graph Theory is an area of discrete mathematics that deals with the study of graphs and the field of graph labeling is a prominent research area in Graph Theory [8]. One of the significant areas in graph labeling is *prime labeling* which is used in many applications within Mathematics as well as in several areas of Computer Science and Communication Networks such as coding theory, circuit design, communication network addressing, and data base management etc. In Mathematics, prime graphs provide an idea to study prime numbers from a different view. Further, there are considerable numbers of open problems and literatures are available for prime labeling of various types of graphs. Our previous researches are focused on the prime labeling for Tripartite graphs, Roach graph, Scorpion graph and Crab graph [2],[3],[7]. The present work is aimed to investigate some new results on prime labeling for a special graph called *Circular Ladder graph* with n number of vertices in one cycle where n is even.

We initiate with a simple finite connected undirected graph $G = (V(G), E(G))$ with $2n$ number of vertices. We provide brief summary of definitions and theorems which are necessary for the present investigations.

Definition 1 (*Ladder graph*)
Let P_n denotes the path on n vertices, then the Cartesian product of $P_n \times P_2$, where $n \geq 2$, is called a ladder graph.

Definition 2 (*Circular Ladder graph* (CL_n))
Circular ladder graph is a simple graph obtained by using Cartesian product of cycle graph C_n with n vertices and path graph P_2, and is denoted by CL_n (i.e $C_n \times P_2 = CL_n$). This is isomorphic to the graph obtained by joining the end vertices of the ladder by two new edges in cyclic form.
(Figure 1)

Definition 3 (*Prime Labeling*)
A graph $G = (V(G), E(G))$ with $|V(G)|$ vertices is said to have *prime labeling* if there exist a bijective mapping $f : V(G) \rightarrow \{1,2,3, \ldots, |V(G)|\}$ such that $\gcd(f(u), f(v)) = 1$ for each edge $e = uv$ in $E(G)$. Such a graph is called a *Prime graph*.

Theorem 1. [1]
If $n + 1$ is prime, then $P_n \times P_2$ has a prime labeling. Moreover, this prime labeling can be realized with top

row labels from left to right, $1, 2, \ldots, n$, *and bottom row labels from left to right,*
$n + 2,\ n + 3,\ \ldots,\ 2n,\ n + 1.$

Theorem 2. [1]
$P_n \times P_2$ has a consecutive cyclic prime labeling with the value 1 *assigned to the vertex* u_1 *if and only if* $2n + 1$ *is prime.*

II. MATERIAL AND METHODS

In our research, circular ladder graph is considered for n number of vertices when n is an even integer and the existence of prime labeling of circular ladder graphs were given by the following theorem.

Theorem 01
If n is an even number and $n + 1$ is prime, then CL_n has a prime labeling.

Proof:
If n is an even number and $n + 1$ is prime, then CL_n has a prime labeling with clockwise direction for both outer and inner cyclic graphs.

To prove this theorem, we use the Theorem 1 in [1] to prove the ladder graphs has the prime labeling. Following figure shows the vertex labeling method of the ladder graph with n levels.

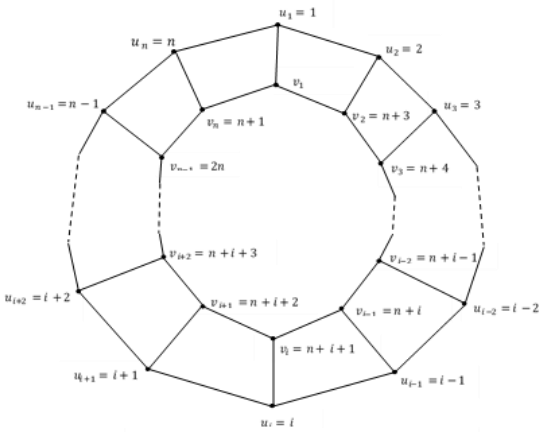

Figure 1: The graph CL_n with vertex labeling

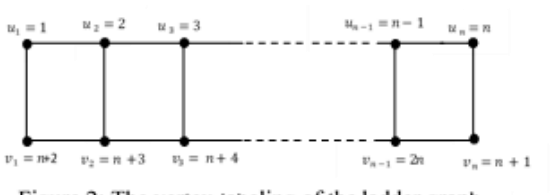

Figure 2: The vertex labeling of the ladder graph

Joining pair of vertices (u_1, u_n) by an edge and the pair of vertices (v_1, v_n) by an edge on the ladder graph (Fig.2) we can construct the graph CL_n which is called the circular ladder graph with $2n$ vertices. Earlier, A. H. Berliner at el. has proved that the above ladder graph has prime labeling in [1].

Hence, it is sufficient to prove that the $\gcd(u_1, u_n) = 1$ and $\gcd(v_1, v_n) = 1$. Since, $\gcd(u_1, u_n) = \gcd(1, n) = 1$ and $\gcd(v_1, v_n) = \gcd(n + 2, n + 1) = 1$ (Consecutive positive numbers), CL_n graph has a prime labeling when n is an even number and $n + 1$ is prime.

Example 1
When $n = 6$, $n + 1 = 6 + 1 = 7$ (Prime), by Theorem 1,

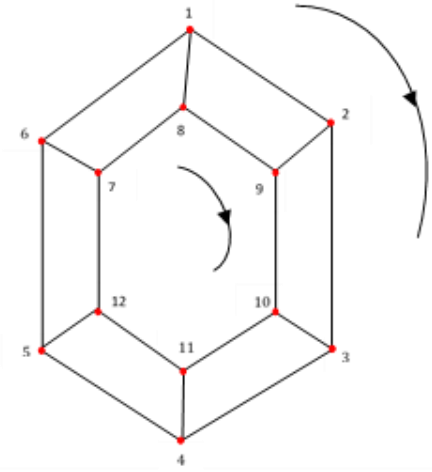

Figure 3: Prime labeling of CL_6 using the theorem 1.

Theorem 02
If n is an even number and $2n + 1$ is a prime, then CL_n has a prime labeling.

Proof:
First, we label vertices in the graph CL_n using integers $1, 2, \ldots, 2n$ with the vertex $u_1 = 1$ and labeling of outer

cyclic graph in clockwise direction and inner cyclic graph in anticlockwise direction.

According to theorem 2 given in [1], the ladder graphs have another prime labeling method and using that to obtain prime labeling of the CL_n, we have to prove that $\gcd(u_1, u_n) = 1$ and $\gcd(v_1, v_n) = 1$.

Clearly, $\gcd(u_1, u_n) = \gcd(1, n) = 1$ and $\gcd(v_1, v_n) = \gcd(2n, n+1) = \gcd(n-1, n+1) = 1$ (consecutive odd numbers)
Hence, CL_n is a prime graph.

Example 2
When $n = 8$; $2n + 1 = 16 + 1 = 17$ (Prime) by Theorem 2,

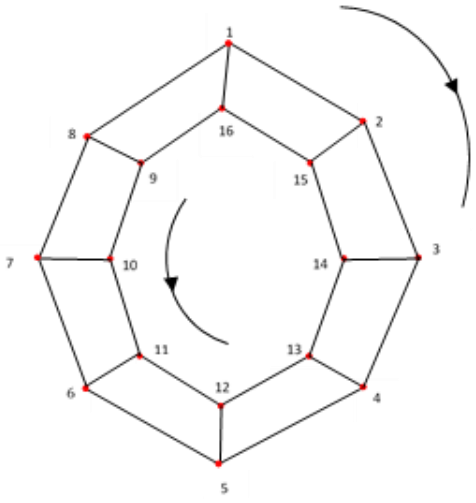

Figure 4: Prime labeling of CL_8 using the theorem 2

III. RESULTS AND DISCUSSION

It has been shown that the graph of CL_6 has prime labeling when $n = 6$ ($n + 1$ is a prime) and the graph of CL_8 has prime labeling when $n = 8$ ($2n + 1$ is a prime).

Further, this work could be generalized as follows:

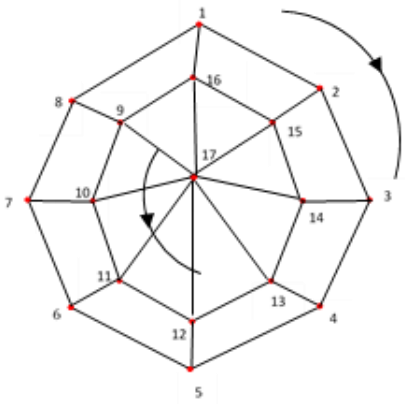

Figure 5: Prime labeling of CL_8 with center vertex.

Remark:
Introducing a new vertex to the center and joining it to all the vertices in the inner circle of CL_n, can be obtained a concentric double wheel graph and labeling the new vertex as $2n + 1$ which is the largest prime in the set. Since $2n + 1$ is a prime number, $\gcd(v_i, 2n + 1)$ for $i = 1, 2, \ldots, n$ for each of the corresponding vertices in the inner circle of CL_n.

CONCLUSION

Prime labeling is the most fascinating area of the graph labeling with various applications. It has been shown that circular ladder graphs are prime graphs under certain conditions according to two theorems of ladder graphs in [1]. These prime labeling methods can be used to study the relationship between graph structures and prime numbers. As a future work, we are planning to generalize these results when CL_n graph has more cycles.

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