An Alternative Method of Constructing Skew-Hadamard Matrices of Order 2^n

A.P. BATUWITA[1], N.T.S.G. GAMACHCHIGE[2], P.G.R.S. RANASINGHE[3], A.A.I. PERERA[4]
[1, 2] Department of Science and Technology, University of Uva Wellassa, Sri Lanka.
[3,4] Department of Mathematics, Faculty of Science, University of Peradeniya, Sri Lanka.

Abstract- In this paper we proposed an alternative method of constructing skew-Hadamard matrices of order 2^n. The doubling construction method can be used to construct an infinite number of skew-Hadamard matrices of order $2^n (n \geq 2)$.

Indexed Terms- Hadamard matrix, skew-Hadamard matrix

I. INTRODUCTION

A matrix H of order n with entries ± 1 and satisfying $HH^T = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n is called a Hadamard matrix of order n. It is well known that the order of a Hadamard matrix is 1,2 or a multiple of 4[1]. Hadamard matrices have been used in many applications in different fields, such as Olivia MFSK, Balanced repeated replication, Hadamard transform, etc.

Hadamard matrices can be constructed in many ways. A new Hadamard matrix can be obtained from a known Hadamard matrix using the method known as the Sylvester construction. If H_n is an $n \times n$ Hadamard matrix, then a $2n \times 2n$ matrix H_{2n} can be defined by $H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$. Among these, Skew-Hadamard matrices are of special interest. If H is a Hadamard matrix and $H + H^T = 2I_n$, then H is called a Skew-Hadamard matrix. Any square matrix H of order n is symmetric if $H = H^T$ and skew symmetric if $H + H^T = 0$. Skew-Hadamard matrices are used to construct several combinatorial objects, such as association schemes, self-dual codes, strongly regular graphs etc. [2].There are various methods to construct skew-Hadamard matrices. But there are many of orders of skew-Hadamard matrices those have not been constructed yet. Though the constructions of most Hadamard matrices are known, not all of those give skew-Hadamard matrices [3].

Goethals and Seidel proposed a new method to construct skew-Hadamard matrices using four square circulant matrices [4]. Wallis constructed skew-Hadamard matrices of order 92 and he proposed the doubling construction method to construct these matrices[5]. In 1972, Wallis and Whiteman proposed a method to construct unknown skew-Hadamard matrices using four incidence matrices [6].

In this paper we propose an alternative method to construct skew-Hadamard matrices of order 2^n.

II. MATERIAL AND METHODS

Consider a 2×2 matrix $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then H_2 satisfies $H_2 {H_2}^T = 2I_2$ and
$H_2 + {H_2}^T = 2I_2$. Therefore, H_2 is both Hadamard and skew-Hadamard.
Let's define H_4 as $H_4 = \begin{bmatrix} H_2 & H_2 \\ -{H_2}^T & {H_2}^T \end{bmatrix}$. Then, H_4 is both Hadamard and skew-Hadamard.
This doubling construction can be used to construct skew-Hadamard matrices of order 2^n.

III. RESULTS AND DISCUSSION

Consider the 2×2 matrix $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Let $H_4 = \begin{bmatrix} H_2 & H_2 \\ -{H_2}^T & {H_2}^T \end{bmatrix} = \begin{bmatrix} 1 & - & 1 & - \\ 1 & 1 & 1 & 1 \\ - & - & 1 & 1 \\ 1 & - & - & 1 \end{bmatrix}$ and

${H_4}^T = \begin{bmatrix} 1 & 1 & - & 1 \\ - & 1 & - & - \\ 1 & 1 & 1 & - \\ - & 1 & 1 & 1 \end{bmatrix}$

(Here, we denote -1 by – sign.)

Then, $H_4{H_4}^T = 4I_4$ and $H_4 + {H_4}^T = 2I_4$.
Therefore, H_4 is both Hadamard and skew-Hadamard.
Let

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ -{H_4}^T & {H_4}^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & - & - & 1 & 1 \\ 1 & - & - & 1 & 1 & - & - & 1 \\ - & - & 1 & - & 1 & 1 & - & 1 \\ 1 & - & 1 & 1 & - & 1 & - & - \\ - & - & - & 1 & 1 & 1 & 1 & - \\ 1 & - & - & - & - & 1 & 1 & 1 \end{bmatrix} \text{ and}$$

$${H_8}^T = \begin{bmatrix} 1 & 1 & - & 1 & - & 1 & - & 1 \\ - & 1 & - & - & - & - & - & - \\ 1 & 1 & 1 & - & 1 & 1 & - & - \\ - & 1 & 1 & 1 & - & 1 & 1 & - \\ 1 & 1 & - & 1 & 1 & - & 1 & - \\ - & 1 & - & - & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & - & 1 & 1 \\ - & 1 & 1 & 1 & 1 & - & - & 1 \end{bmatrix}$$

Then, $H_8{H_8}^T = 4I_8$ and $H_8 + {H_8}^T = 2I_8$.
Therefore, H_8 is both Hadamard and skew-Hadamard.
Generalization of this construction is given by the following Theorem.

Theorem:
Let $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ be a skew Hadamatd matrix of order 2. Then,

$H_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H^T_{2^{n-1}} & H^T_{2^{n-1}} \end{bmatrix}$ is a skew-Hadamard matrix of order 2^n.

Proof
Since $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is a skew Hadamard matrix of order 2, H_2 satisfies $H_2{H_2}^T = 2I_2$ and $H_2 + {H_2}^T = 2I_2$.

Consider $H_4 = \begin{bmatrix} H_2 & H_2 \\ -{H_2}^T & {H_2}^T \end{bmatrix}$. Then,

${H_4}^T = \begin{bmatrix} {H_2}^T & -H_2 \\ {H_2}^T & H_2 \end{bmatrix}$.

Note that

$$H_4{H_4}^T = \begin{bmatrix} H_2 & H_2 \\ -{H_2}^T & {H_2}^T \end{bmatrix}\begin{bmatrix} {H_2}^T & -H_2 \\ {H_2}^T & H_2 \end{bmatrix}$$

$$= \begin{bmatrix} H_2{H_2}^T + H_2{H_2}^T & -H_2H_2 + H_2H_2 \\ -H_2{H_2}^T + H_2{H_2}^T & H_2{H_2}^T + H_2{H_2}^T \end{bmatrix}$$

$$= \begin{bmatrix} 2I_2 + 2I_2 & 0 \\ 0 & 2I_2 + 2I_2 \end{bmatrix} = \begin{bmatrix} 4I_2 & 0 \\ 0 & 4I_2 \end{bmatrix} = 4I_4$$

and

$$H_4 + {H_4}^T = \begin{bmatrix} H_2 & H_2 \\ -{H_2}^T & {H_2}^T \end{bmatrix} + \begin{bmatrix} {H_2}^T & -H_2 \\ {H_2}^T & H_2 \end{bmatrix}$$

$$= \begin{bmatrix} H_2 + {H_2}^T & H_2 - H_2 \\ -{H_2}^T + {H_2}^T & {H_2}^T + H_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2I_2 & 0 \\ 0 & 2I_2 \end{bmatrix} = 2\begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} = 2I_4.$$

i.e., $H_4{H_4}^T = 4I_4$ and $H_4 + {H_4}^T = 2I_4$. Here "0" denotes the zero matrix of the same order.
Therefore, H_4 is Hadamard and skew-Hadamard.

Now consider, $H_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H^T_{2^{n-1}} & H^T_{2^{n-1}} \end{bmatrix}$.

Then, $H^T_{2^n} = \begin{bmatrix} H^T_{2^{n-1}} & -H_{2^{n-1}} \\ H^T_{2^{n-1}} & H_{2^{n-1}} \end{bmatrix}$.

Note that

$$H_{2^n}H^T_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H^T_{2^{n-1}} & H^T_{2^{n-1}} \end{bmatrix}\begin{bmatrix} H^T_{2^{n-1}} & -H_{2^{n-1}} \\ H^T_{2^{n-1}} & H_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} H_{2^{n-1}}H^T_{2^{n-1}} + H_{2^{n-1}}H^T_{2^{n-1}} & -H_{2^{n-1}}H_{2^{n-1}} + H_{2^{n-1}}H_{2^{n-1}} \\ -H_{2^{n-1}}H^T_{2^{n-1}} + H_{2^{n-1}}H^T_{2^{n-1}} & H_{2^{n-1}}H^T_{2^{n-1}} + H_{2^{n-1}}H^T_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} 2^nI_{2^{n-1}} & 0 \\ 0 & 2^nI_{2^{n-1}} \end{bmatrix} = 2^n\begin{bmatrix} I_{2^{n-1}} & 0 \\ 0 & I_{2^{n-1}} \end{bmatrix} \text{ and}$$

$$H_{2^n} + H^T_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H^T_{2^{n-1}} & H^T_{2^{n-1}} \end{bmatrix} + \begin{bmatrix} H^T_{2^{n-1}} & -H_{2^{n-1}} \\ H^T_{2^{n-1}} & H_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} H_{2^{n-1}} + H^T_{2^{n-1}} & H_{2^{n-1}} - H_{2^{n-1}} \\ -H^T_{2^{n-1}} + H^T_{2^{n-1}} & H^T_{2^{n-1}} + H_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} 2I_{2^{n-1}} & 0 \\ 0 & 2I_{2^{n-1}} \end{bmatrix} = 2\begin{bmatrix} I_{2^{n-1}} & 0 \\ 0 & I_{2^{n-1}} \end{bmatrix}.$$

i.e., $H_{2^n}H^T_{2^n} = 2^nI_{2^n}$ and $H_{2^n} + H^T_{2^n} = 2I_{2^n}$
Therefore, H_{2^n} is both Hadamard and skew-Hadamard.

Note:
Consider a 2×2 skew-Hadamard matrix $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = {H_2}^T$.

Then $A_{2^n} = \begin{bmatrix} A^T_{2^{n-1}} & A^T_{2^{n-1}} \\ -A_{2^{n-1}} & A_{2^{n-1}} \end{bmatrix}$ is both Hadamard and skew-Hadamard of order 2^n.

There are various methods to construct skew-Hadamard matrices. In this research, an alternative method was proposed. The doubling construction can be used to construct an infinite number of skew-Hadamard matrices of order $2^n (n \geq 2)$.

CONCLUSION

A special emphasis can be given to Skew-Hadamard matrices among other Hadamard matrices. First, 2×2 skew-Hadamard matrices were considered. Using the proposed doubling construction method, our main result shows that skew-Hadamard matrices of orders $4,8,16,32,64, \dots, 2^n$ $(n \geq 2)$ can be constructed. Further, this construction was illustrated with some examples. As a future work, we are planning to implement a computer programme to construct large skew-Hadamard matrices of order 2^n and unknown skew-Hadamard matrices.

REFERENCES

[1] R. Paley, "On orthogonal matrices," Journal of Mathematics and Physics, pp. 311-320, 1933.

[2] Hanaki, H. Kharaghani, A. Mohammadian and B. Tayfeh-Rezaie, "Classification of skew-Hadamard matrices of order 32 and association schemes of order 31," Journal of Combinatorial Design, pp. 421-427, 2020.

[3] C. Koukouvinos and S. Stylianou, "On skew-Hadamard matrices," ScienceDirect, p. 2723–2731, 2008.

[4] J. Goethals and J. Seidel, "A Skew Hadamard Matrix of Order 36.," Journal of the Australian Mathematical Society, pp. 343 - 344, 1970.

[5] J. Seberry, "A skew-Hadamard matrix of order 92," Bulletin of the Australian Mathematical Society, pp. 203-204, 1971.

[6] J. Wallis and A. Whiteman, "Some classes of Hadamard matrices with constant diagonal," Bulletin of the Australian Mathematical Society, pp. 233-249, 1972.