# On the Duals of Some Banach Spaces

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***Abstract-*** ***The main purpose of this paper is to establish the reflexive space of $\ell^p$. We establish whether the linear functional $F_x : (\ell^p)^* \to C$ defined by $F_x(f) = f(x)$ is bounded $\forall f \in (\ell^p)^*$ i.e. $F_x \in L^p$ and $\|F_x\| = \|x\|$ and whether the Canonical mapping given by $C : \ell^p \to L^p$ is linear and bijective $\forall x \in \ell^p$ and $F_x \in L^p$.***

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***Indexed Terms- Dual Spaces, Reflexive Spaces.***

## I. INTRODUCTION

We shall consider a normed space X, its dual $X^*$ and the dual of $X^*$ of which we represent by $(X^*)^*$. Now $X^{**}$ is some times called the second dual of $X$. We define the functional $g_x$ on $X^*$ by choosing a fixed $x \in X$ and setting $g_x(f) = f(x)$ $\forall f \in X^*$. The linear functional $f$ is bounded and we show that $g_x$ is a bounded linear functional on $X^*$.

## II. DUAL SPACES

2.1 Definition
If $X$ is a linear space, the space of functions on $X$ is called continuous linear functionals. They are also called dual spaces.

2.2 Definition
If $X$ is normed linear space, the set of all bounded linear functionals on $X$ denoted by $X^*$ is also normed where the norm is defined as:

$$\|f\| = \sup\left( \frac{|f(x)|}{\|x\|} : x \in X, x \neq 0 \right)$$

2.3 Definition
Let $X$ and $Y$ be linear spaces. Then a function $T : X \to Y$ is called a linear operator if and only if.

=

$$T(\lambda x_1 + \mu x_2) = \lambda T(x_1) + \mu T(x_2), \forall x_1, x_2 \in X, \lambda, \mu \in$$
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2.4 Definition
An isomorphism of normed space X onto a normed space Y is a bijective linear operator $T : X \to Y$ which preserves the norm i.e., $\|Tx\| = \|x\|, \forall x \in X$. X is said to be isomorphic to Y.

## III. REFLEXIVE SPACES

We shall consider a normed space $X$, its dual $X^*$ and the dual of $X^*$ of which we represent by $(X^*)^*$ Now $X^{**}$ is some times called the second dual of $X$ We define the functional $g_x$ on $X^*$ by choosing a fixed $x \in X$ and setting $g_x(f) = f(x)$ $\forall f \in X^*$. The linear functional $f$ is bounded and we show that $g_x$ is a bounded linear functional on $X^*$ as we shall see below:

3.1 Lemma
Let $X$ be a normed linear space and $x \in X$, then the linear functional $g_x(f) = f(x)$ is bounded $\forall f \in X^*$ i.e. $g_x \in X^{**}$ and $\|g_x\| = \|x\|$.

Proof
Linearity

$$g_x(\alpha_1 f_1 + \alpha_2 f_2) = (\alpha_1 f_1 + \alpha_2 f_2)(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) = \alpha_1 g_x(f_1) + \alpha_2 g_x(f_2)$$

$\forall f_1, f_2 \in X^*, \alpha_1, \alpha_2 \in \mathbf{K}$ and $g_x \in X^{**}$

$(\alpha g_x)(f) = (\alpha f)(x) = \alpha f(x) = \alpha g_x(f).$

$\forall g_x \in X^{**}, f \in X^*, \alpha \in \mathbf{K}$

Moreover

$$\|g_x\| = \sup\left\{\frac{|g_x(f)|}{\|f\|} : f \in X^*, f \neq 0\right\} = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\} = \|x\|.$$

Now for each $x \in X$, there corresponds a unique bounded linear functional $g_x(f) = f(x).$ This defines a mapping.

$C : X \to X^{**}$ such that $C(x) = g_x$.

$C$ is called the canonical mapping of $X$ on to $X^{**}$

We shall show that $C$ is linear and bijective.

3.2 Lemma

The canonical mapping $C$ given by:

$$X \to X^{**}$$
$$x \to g_x$$

is linear and bijective of a normed space $X$ on to the normed space $R(C)$ where $R(C)$ is the range of $C$.

Proof

Linearity of $C$ is clear, since we have

$g_{\alpha x+\beta y}(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_y(f)$

$\forall x, y \in X$ and $\alpha, \beta \in \mathbf{K}$

Thus $C(\alpha x + \beta y) = \alpha C(x) + \beta C(y)$

$(\alpha g_x)(f) = (\alpha f)(x) = \alpha f(x) = \alpha g_x(x).$

Thus $C(\alpha x) = \alpha C(x).$

If $C(x) = C(y),$ then $g_x = g_y$, so $x - y$ .Thus C is 1-1.

If $g_x \in X^{**}$ then $C(x) = g_x$, where $x \in X$. Thus, C is onto $X^{**}$.

This shows that C is linear and bijective.

3.3 Definition

A normed space $X$ is said to be embeddedable in a normed space $Z$ if $X$ is isomorphic to a sub space $Z$.

Lemma 1.1 And 1.2 shows that $X$ is embeddable in $X^{**}$ and C is also called the canonical embedding of $X$ into $X^{**}$.

3.4 Definition

A normed space $X$ is said to be reflexive if the range $R(C)$ of $C$ is $X^{**}$ where $C$ is the canonical mapping of $X$ onto $X^{**}$. If $X$ is reflexive, it is isomorphic with $X^{**}$ space

## IV. THE DUAL OF $\ell^p$ SPACE.

Let $\ell^p(0 \le p \le \infty)$ be a linear space and $(\ell^p)^* = \ell^q$, the set of linear functionals on $\ell^p$ where $\frac{1}{p} + \frac{1}{q} = 1$ $\ell^q$ is the dual of $\ell^p$.

If $x \in \ell^p$ then $\|x\| = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$ and if $f \in (\ell^p)^*$ then $\|f\| = \left(\int_x |f|^p\right)^{\frac{1}{p}}$

The schauder basis of $\ell^p$ is $e_k$, where

$$e_k = \partial_{i.k} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}.$$

$\|e_k\| = 1, \forall \kappa = 1,2,\ldots$

We define

$$x = \sum_{k=1}^{\infty} x_k e_k, \forall x = (x_1, x_2, \ldots) \in \ell^p$$

and $f(x) = \sum_{k=1}^{\infty} x_k \eta_k$ where $\eta_k = f(e_k)$.

The following theorem illustrates that $\ell^q$ is the dual of $\ell^p$.

4.1 Lemma

Let $\ell^p$ be a linear space and $x \in \ell^p$, then the linear functional $f \in (\ell^p)^*$ is bounded i.e. $\|f\| = \|x\|$.

Proof

Isometric

Let $x_n = x_k^{(n)}$ where

$$x_k^{(n)} = \begin{cases} \frac{|\eta_k|^q}{\eta_k}, k \le n \\ 0, k > n \end{cases} \text{ where } \eta_k = f(e_k)$$

It follows that,

$$f(x_n) \le \|f\|\|x_n\| = \|f\|\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p} = \|f\|\left(\sum_{k=1}^{n}|\eta_k|^q\right)^{\frac{1}{p}}$$

$$\left\{|x_k|^p = \frac{|\eta_k|^{pq}}{|\eta_k|^p} = |\eta_k|^q, \forall q = qp - p\right\}$$

But $f(x_n) = \sum x_k^{(n)}\eta_k = \sum_{k=1}^{n}|\eta_k|^q$

Therefore $\sum_{k=1}^{n}|\eta_k|^q \le \|f\|\left(\sum^{n}|\eta_k|^q\right)^{\frac{1}{p}}$

So that $\left(\sum_{k=1}^{n}|\eta_k|^q\right)^{\frac{1}{q}} \le \|f\|$

$$\Rightarrow n = (n_k) \in \ell^q, so\|n\| \le \|f\|.$$

Conversely,

Suppose $n = (n_n) \in \ell^q$ is given.

Define $f$ on $\ell^p$ by

$$f(x) = \sum_{k=1}^{n}\eta_k x_k, x = (x_n) \in \ell^p$$

By Holders inequality,

$$|f(x)| \le \left(\sum_{k=1}^{n}|\eta_k|^q\right)^{1/q}\left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p} = \|\eta\|\|x\|.$$

Therefore

$$\|f\| \le \|\eta\| \Rightarrow f \in \ell^q$$

Since $f(e_k) = \eta_k$ and $\|\eta\| \le \|f\|$

$$\Rightarrow \|f\| = \|\eta\|.$$

If $x = (\eta_1, \eta_2 .....\eta_n)$, then it follows that $\|f\| = \|x\|$.

Now there corresponds a unique bounded linear functional $f \in (\ell^p)^*$ given by

$f(x) = \sum_{k=1}^{\infty} x_k\eta_n$. This defines a mapping $T : (\ell^p)^* \to \ell^q$.

4.2 Lemma

The mapping $T : (\ell^p)^* \to \ell^q$ is linear and bijective.

Proof

T is linear

Let $f, g \in (\ell^p)^*, f(\ell_k) = \eta_k$ and $g(e_k) = \eta_k, \forall k \in \mathrm{N}$

Then

$Tf = \alpha_1, \alpha_2, \alpha_3........$ and

$Tg = (\alpha_1^{'}, \alpha_2^{'}, \alpha_3^{'}...........) \in \ell^q$ .

Therefore

$f + g \in (\ell^p)^*$ and $(f + g)(e_k)$

$= f(e_k) + g(e_k) = \alpha_k + \alpha_k^{'}$

Therefore

$$T(f + g) = (\alpha_1 + \alpha_1^{'}, \alpha_2 + \alpha_2^{'}. + .....)$$

$$= (\alpha_1, \alpha_2 .....) + (\alpha_1^{'}, \alpha_2^{'} .......) = Tf + Tg.$$

Likewise

$$T(\lambda f) = (\lambda\alpha_1, \lambda\alpha_2, ...) = \lambda(\alpha_1, \alpha_2, ..). = \lambda Tf.$$

T is 1-1

Let $f, g \in (\ell^p)^*$ and $Tf = Tg$, then $Tf - Tg = 0$ (since T is linear).

But since T is an isometry. $\|T(f-g)\| = \|f-g\| = \|0\| = 0$

So that $\|f-g\| = 0 \Rightarrow f-g=0$

T is onto

If $(\beta_1\beta_2........) \in \ell^q$ then there is a g $\in (\ell^p)^* : Tg = (\beta_1, \beta_2, ....)$

i.e. $g(e_k) = \beta_k, \forall k \in K$

Let $x = (\alpha_1, \alpha_2, ......) \in \ell^p$ and define $g : \ell^p \to$ K by

$g(x) = \sum_{k=1}^{\infty} \alpha_k \beta_k$ then g is well defined and linear.

Let $x = (\alpha_1, \alpha_2, ..........), y = (\lambda_1, \lambda_2, ....) \in \ell^p$, then $x + y = (\alpha_1 + \lambda_1, \alpha_2 + \lambda_2, ....)$

Therefore

$$g(x+y) = \sum_{k=1}^{\infty} (\alpha_k + \lambda_k)\beta_k = \sum_{K=1}^{\infty} \alpha_K \beta_K = g(x) + g(y)$$

Therefore

$T : (\ell^p)^* \to \ell^q$ is linear, isometry, 1-1 and on to. Hence $(\ell^p)^*$ is congruent to $\ell^q$.

## V. METHODS

To establish the reflexive space of $\ell^p$, we shall use lemma 3.1, 3.2, 4.1 and 4.2 on boundedness, linearity and bijective mappings.

## RESULTS AND DISCUSSIONS

We define the functional

$F_x$ on $(\ell^p)^*$ by choosing a fixed $x \in \ell^p$ and setting $F_x(f) = f(x)$, for all $f \in (\ell^p)^*$. The linear functional $f$ is bounded and we show that $F_x$ is bounded linear functional on $(\ell^p)^*$.

6.1 Proposition

Let $x \in \ell^p$, then the linear functional $F_x : (\ell^p)^* \to$ C defined by

$F_x(f) = f(x), \forall f \in (\ell^p)^*$ is bounded

i.e. $F_x \in L^p$ and $\|F_x\| = \|x\|$.

Proof

Linearity

$F_x(\alpha_1 f_1 + \alpha_2 f_2) = (\alpha_1 f_1 + \alpha_2)(x) =$

$\alpha_1 f_1(x) + \alpha_2 f_2(x) = \alpha_1 F_x(f_1) + \alpha_2 f_x(f_2)$

$\forall f_1, f_2 \in (\ell^p)^*, \alpha_1, \alpha_2 \in K.$

$(\alpha F_x)(f) = (\alpha f)(x) = \alpha F_x(f), \forall \alpha \in \kappa.$

Isometric

$$\|F_x\| = \sup\left\{\frac{|F_x(f)|}{\|f\|} : f \in (\ell^p)^*, f \neq o\right\}$$

$$= \sup\left\{\frac{|f(x)|}{\|f\|} : f \in (\ell^p)^*, f \neq 0\right\} = \|x\|, \forall x \in \ell^p$$

i.e. $\|F_x\| = \|x\|$.

Now for each $x \in \ell^p$ there corresponds a unique bounded linear functional $F_x(f) = f(x), \forall f \in (\ell^p)^*$.

This defines a mapping $C : \ell^p \to L^p$ such that $C(x) = F_x$. C is called the canonical mapping of $\ell^p$ on to $L^p$.

6.2 Proposition

The canonical mapping C given by $C : \ell^p \to L^p$ is linear and bijective $\forall x \in \ell^p$ and $F_x \in L^p$.

Proof

Linearity of C

$F_{x+y}(f) = f(x+y) = f(x) + f(y)$

$= F_x(f) + Fy(f)$

$\forall x, y \in \ell^p$.

$(\alpha F_x)(f) = (\alpha f)(x) = \alpha F_x(f) \forall \alpha \in \kappa$

C is 1-1

For $x \in \ell^p$, define $C(x) = F_x$, this gives a mapping $C : \ell^P \to L^p$

If $C(x) = C(y)$, then $F_x = F_y$, so $x = y$ .Thus C is 1-1

C is onto.

If $F_x \in L^p$ then $C(x) = F_x$, where $x \in \ell^P$. Thus C is onto $L^P$.

## CONCLUSIONS AND RECOMMENDATIONS

Both the dual and reflexive spaces of $\ell^p$ have drawn substantial interest in this study where,

$$\frac{1}{p} + \frac{1}{q} = 1$$

We have established that:

i. The linear functional $F_x : (\ell^p)^* \to \mathbf{C}$ defined by $F_x(f) = f(x)$ is bounded $\forall f \in (\ell^p)^*$ i.e. $F_x \in L^p$ and $\|F_x\| = \|x\|$

ii. The canonical mapping C given by $C : \ell^P \to L^p$ is linear and bijective $\forall x \in \ell^p$ and $F_x \in L^P$.

Therefore (i) and (ii) implies that $\ell^p$ space is reflexive and is isomorphic to $L^p$ .We recommend that effort be directed towards establishing duality among other Banach spaces.

## REFERENCES

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