On Class (N+K)-Power (BD) Operators

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***Abstract-** In this paper, we introduce the class of (n+k)-power (BD) operators acting on a complex Hilbert space H. An operator if $T \in B(H)$ is said to belong to class(n+k)-power (BD) if $T^{*2}(T^D)^2$ commutes with $(T^*T^D)^2$ equivalently $[T^{*2}(T^D)^2, (T^*T^D)^2] = 0$. We investigate the properties of this class and we also analyze the relation of this class to(n+k)-power D-operator*

***Indexed Terms-** D-operator, Normal, N Quasi D-operator, complex symmetric operators, n-power D-operator, (BD) operators.*

I. INTRODUCTION

Throughout this paper, H denotes the usual Hilbert space over the complex field and B(H) the Banach algebra of all bounded linear algebra on an infinite dimensional separable Hilbert space H. A bounded linear operator T is said to be in class (Q) if $T^{*2}T^2 = (T^*T)^2$(2). This was later extended into other classes like class (Q) (2), n-power class (Q) if $T^{*2}T^{2n} = (T^*T^n)^2$(3), quasi-M class (Q) and (α, β)-class (Q) we refer the reader to (6) for more. An operator $T \in B(H)$ is said to belong to class (BQ) if $T^{*2}T^2(T^*T)^2 = (T^*T)^2T^{*2}T^2$ An operator $T \in B(H)$ is said to be D-operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$ where T^D is the Drazin inverse of T (1).Wanjala Victor and A.M. Nyongesa later extended this to N Quasi D-operator (3), a bounded linear operator T is said to be N Quasi D-operator if $T(T^{*2}(T^D)^2 = N(T^*T^D)^2T$ where N is a bounded linear operator. A bounded linear operator T is said to belong to class (BD) provided $T^{*2}(T^D)^2$ commutes with $(T^*T^D)^2$ where T^D is the Drazin inverse of T. Let H be a Hilbert space, then a conjugation on H is an anti-linear operator C from H onto itself such that the following is satisfied Cξ, Cζi= hζ, ξifor every $\xi, \zeta \in H$ and $C^2 = I$. We say that T is complex symmetric if $T = CT^*C$.

II. MAIN RESULTS

- Theorem 1. Let $T \in B(H)$ be such that $T \in$(n+k)-power (BD), then the following are also true for(n+k)-power (BD);

i. λT for any real λ
ii. Any $S \in B(H)$ that is unitarily equivalent to T.
iii. The restriction T-M to any closed subspace M of H.

Proof.

i. The proof is trivial.
ii. Let $S \in B(H)$ be unitarily equivalent to T, then there exists a unitary operator $U \in B(H)$ with $S = U^*TU$ and $S^* = U^*T^*U$. Since $T \in$(n+k)-power(BD), we have; $T^{*2}(T^D)^{2(n+k)}(T^*(T^D)^{n+k})^2 = (T^*(T^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}$, hence $S^{*2}(S^D)^{2(n+k)}(S^*(S^D)^{n+k})^2 = UT^{*2}U^*U(T^D)^{2(n+k)}U^*(UT^*U^*U(T^D)^{n+k}U^*)^2 = UT^{*2}U^*U^*(T^D)^{2(n+k)}U^*UT^*U^*UT^*U^*U(T^D)^{n+k}U^*U(T^D)^{n+k}U^*$
$= UT^{*2}(T^D)^{2(n+k)}(T^*(T^D)^{n+k})^2U^*$
$= U(T^*(T^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}U^*$
and
$(S^*(S^D)^{n+k})^2S^{*2}(S^D)^{2(n+k)} = (UT^*U^*U(T^D)^{n+k}U^*)^2UT^{*2}U^*U(T^D)^{2(n+k)}U^*$
$= UT^*U^*U(T^D)^{n+k}U^*UT^*U^*U(T^D)^{n+k}U^*UT^{*2}U^*U(T^D)^{2(n+k)}U^*$
$= UT^*(T^D)^{n+k}T^*(T^D)^{n+k}T^{*2}T^{*2}U^*$
$= U(T^*(T^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}U^*$
Hence S is unitarily equivalent to T.
iii. If T is in class(n+k)-power (BD), then;
$T^{*2}(T^D)^{2(n+k)}(T^*(T^D)^{n+k})^2 = (T^*(T^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}$.
Hence;
$(T/M)^{*2}(((T/M)^D)^{n+k})^2\{(T/M)^*((T/M)^D)^{n+k}\}^2$
$=(T/M)^{*2}(((T/M)^D)^{n+k})^2\{(T/M)^*((T/M)^D)^{n+k}\}^2$
$= (T^{*2}/M)((T^D)^{2(n+k)}/M)\{(T^*/M)((T^D)^{n+k}/M)\}\{(T^*/M)((T^D)^{n+k}/M)\}$
$= \{(T^*(T^D)^{n+k})^2/M\}\{T^{*2}(T^D)^{2(n+k)}/M\}$
$= \{(T^*/M)((T^D)^{n+k}/M)\}^2(T/M)^{*2}(((T/M)^D)^{n+k})^2$

Hence $T/M \in$(n+k)-power (BD).

- Theorem 2. If $T \in B(H)$ is an (n+k)-power D-operator, then $T \in$(n+k)-power (BD).

Proof. Suppose T is an (n+k)-power D-operator, then
$T^{*2}(T^D)^{2(n+k)} = (T^*(T^{D)\,n+k})^2$
post multiplying both sides by $T^{*2}(T^D)^{2(n+k)}$;
$T^{*2}(T^D)^{2(n+k)}\,T^{*2}(T^D)^{2(n+k)} = (T^*(T^{D)\,n+k})^2 T^{*2}(T^D)^{2(n+k)}$
$T^{*2}(T^D)^{2(n+k)}\,T^*(T^{D)\,n+k}T^*(T^{D)\,n+k} = (T^*(T^{D)\,n+k})^2 T^{*2}(T^D)^{2(n+k)}$
$T^{*2}(T^D)^{2(n+k)}(T^*(T^{D)\,n+k})^2 = (T^*(T^{D)\,n+k})^2 T^{*2}(T^D)^{2(n+k)}$.

- Theorem 3. Let $S \in$(n+k)-power (BD) and $T \in$(n+k)-power (BD). If both S and T are doubly commuting, thenST is in (n+k)-power (BD). Proof.

$(ST)^{*2}((ST)^D)^{2(n+k)}((ST)^*(ST)^D)^{2(n+k)}$
$=S^{*2}T^{*2}(S^D)^{2(n+k)}(T^D)^{2(n+k)}((ST)^*((ST)^D)^{n+k})((ST)^*((ST)^D)^{n+k})$
$=S^{*2}T^{*2}(S^D)^{2(n+k)}(T^D)^{2(n+k)}((S^*T^*)((ST)^D)^{n+k})((S^*T^*)((ST)^D)^{n+k})$
$=S^{*2}T^{*2}(S^D)^{2(n+k)}(T^D)^{2(n+k)}S^*T^*(S^D)^{n+k}(T^D)^{n+k}S^*T^*(S^D)^{n+k}(T^D)^{n+k}S^*T^*(S^D)^{n+k}(T^D)^{n+k}$
$= S^{*2}T^{*2}(SD)^{2(n+k)}(T^D)^{2(n+k)}S^*(S^D)^{n+k}T^*(T^D)^{n+k}S^*(S^D)^{n+k}T^*(T^D)^{n+k}$
$=T^{*2}(T^D)^{2(n+k)}S^{*2}(S^D)^{2(n+k)}S^*(S^D)^{n+k}S^*(S^D)^{n+k}T^*(T^D)^{n+k}T^*(T^D)^{n+k}$

$=T^{*2}(T^D)^{2(n+k)}S^{*2}(S^D)^{2(n+k)}(S^*(S^D)^{n+k})^2T^*(T^D)^{n+k}T^*(T^D)^{n+k}$
$=T^{*2}(T^D)^{2(n+k)}(S^*(S^D)^{n+k})^2S^{*2}(S^D)^{2(n+k)}T^*(T^D)^{n+k}T^*(T^D)^{n+k}$ (Since $S \in$(n+k)-power(BD)).
$= (S^*(S^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}T^*(T^D)^{n+k}T^*(T^D)^{n+k}S^{*2}(S^D)^{2(n+k)}$
$=(S^*(S^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}(T^*(T^D)^{n+k})^2S^{*2}(S^D)^{2(n+k)}$
$=(S^*(S^D)^{n+k})^2(T^*(T^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}S^{*2}(S^D)^{2(n+k)}$ (Since $T \in$(n+k)-power(BD)).
$=((S^*(S^D)^{n+k})(T^*(T^D)^{n+k}))^2T^{*2}S^{*2}(T^D)^{2(n+k)}(S^D)^{2(n+k)}$
$=((S^*T^*)((S^D)^{n+k}(T^D)^{n+k}))^2S^{*2}T^{*2}(S^D)^{2(n+k)}(T^D)^{2(n+k)}$
$= ((ST)^*((ST)^D)^{n+k})^2(ST)^{*2}(((ST)^D)^{n+k})^2$

Hence $ST \in$(n+k)-power (BD).

- Theorem 4. Let $T \in B(H)$ be a class (n+k)-power (BD) operator such that $T = CT^*C$ with C being aconjugation on H. If C is such that it commutes with $T^{*2}(T^D)^{2(n+k)}$ and $(T^*(T^D)^{n+k})^2$, then T is an (n+k)-powerD- operator.

Proof. Let $T \in$(BD) and complex symmetric, then we have; $T^{*2}(T^D)^{2(n+k)}(T^*(T^D)^{n+k})^2 = (T^*(T^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}$
and $T = CT^*C$.
hence;
$T^{*2}(T^D)^{2(n+k)}(T^*(T^D)^{n+k})^2$
$=(T^*(T^D)^{n+k})^2T^{*2}(T^D)^{2(n+k)}\quad T^{*2}(T^D)^{2(n+k)}C(T^D)^{n+k}CCT^*CC(T^D)^{n+k}CCT^*C$
$= (T^*(T^D)^{n+k})^2C(T^D)^{n+k}CCT^*CC(T^D)^{n+k}CCT^*C.$
$T^{*2}(T^D)^{2(n+k)}C(T^D)^{n+k}T^*(T^D)^{n+k}T^*C = (T^*(T^D)^{n+k})^2C(T^D)^{n+k}T^*(T^D)^{n+k}T^*C$
$T^{*2}(T^D)^{2(n+k)}C(T^D)^{2(n+k)}T^{*2}C = (T^*(T^D)^{n+k})^2CT^*(T^D)^{n+k}T^*(T^D)^{n+k}C$
$T^{*2}(T^D)^{2(n+k)}CT^{*2}(T^D)^{2(n+k)}C = (T^*(T^D)^{n+k})^2C(T^*(T^D)^{n+k})^2C.$
C commutes with $T^{*2}(T^D)^{2(n+k)}$ and $(T^*(T^D)^{n+k})^2$ hence we obtain ;
$T^{*2}(T^D)^{2(n+k)}T^{*2}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2(T^*(T^D)^{n+k})^2$.
which implies;
$T^{*2}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$ and hence T is an (n+k)-power D-operator.

- Definition 5. An operator T is said to be in class (nBD) if $T^{*2}(T^D)^{2n}(T^*(T^D)^n)^2 = (T^*(T^D)^n)^2T^{*2}(T^D)^{2n}$ for a positive integer n.

Theorem 6. Let $T \in B(H)$ be (n+k-1)-D- operator, if T is a complex symmetric operator such that C commutes with $(T^*(T^D)^{n+k})^2$, then T is an (n+k)-power D- operator.

Proof. With T being complex symmetric and (n-1)-D-operator, we have;
$T = CT^*C$ and $T^{*2}(T^D)^{2(n+k-1)} = (T^*(T^D)^{n+k-1})^2$.
We obtain;
$T^{*2}(T^D)^{2(n+k-1)}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k-1})^2(T^D)^2$.
hence;

$T^{*2}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k-1})^2(T^D)^2$.
$T^{*2}(T^D)^{2(n+k)} = T^{*2}(T^D)^{2(n+k-1)}(T^D)^2 = (T^D)^{2(n+k-1)}T^{*2}(T^D)^2$
$T^{*2}(T^D)^{2(n+k)} = (T^D)^{2(n+k-1)}T^*T^*T^DT^D = (T^D)^{2(n+k-1)}CTDCCTDCCT^*CCT^*C = (T^D)^{2(n+k-1)}CTDTDT^*T^*C.$
$=T^{*2}(T^D)^{2(n+k)} = (T^D)^{2(n+k-1)}C(T^D)^2T^{*2}C = (T^D)^{2(n+k-1)}C(T^*(T^D)^{n+k})^2C$

Since C commutes with $(T^*(T^D)^{n+k})^2$ we obtain;

$T^{*2}(T^D)^{2(n+k)} = (T^D)^{2(n+k-1)}(T^*(T^D)^{n+k})^2CC = (T^D)^{2(n+k-1)}T^{*2}(T^D)^2CC = (T^D)^{2(n+k-1)}(T^D)^2T^{*2}CC = T^{*2}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$

Hence T is (n+k)-power D-operator.

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