On (N+K)- Power- D-Operator

WANJALA VICTOR[1], A.M. NYONGESA[2]
[1, 2] *Kibabii University, Bungoma-Kenya.*

Abstract- In this paper, we introduce the class of (N+K)-power -D-operator acting on the Hilbertspace H over the complex plane. A bounded linear operator T is said to be an (N+K)-power -D-operatorif $T^{*2}(T^{D})^{2(n+k)}) = (T^{*}(T^{D})^{n+k})^{2}$ ***for positive integers n and k and where*** T^D ***is the Drazin inverse of T. We investigate thebasic behavior of this class of operator.***

Indexed Terms- Normal operators, D-Operator, Almost Class (Q), quasi -class (Q) operators, N quasi D-operator.

I. INTRODUCTION

H denotes the separable complex Hilbert space in this paper, while B(H) is the usual Banach algebra of all bounded linear operators on H. Let T ∈ B(H), Drazin inverse of T is an operator T^D∈ B(H), such that $TT^D = T^DT$, $T^D = T^DTT^D$ and $T^{k+1}T^D = T^k$ provided it exists. An operator T ∈ B(H) is said to be D-Operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$, this was covered by Abood and Kadhim in (1),N quasi D- Operator was covered by Wanjala Victor and A.M. Nyongesa in (6), a bounded linear operator T is said to be N Quasi D-operator ifT $(T^{*2}(T^D)^2) = N(T^*T^D)^2T$,M Quasi class (Q) if $T(T^{*2}T^2) = M(T^*T)^2T$ (5),class (Q) if $T^{*2}T^2 = (T^*T)^2$ (4), Quasiclass (Q) if $T(T^{*2}T^2) = (T^*T)^2T$, for a bounded linear operator N. Let T = ξ + iζ, with ξ = Re(T) = $\frac{TD+T*}{2}$and ζ = Im(T) = $\frac{TD-T*}{2i}$. We shall simply denote $U^2 = (T^*T^D)^2$ and $V^2 = T^{*2}(T^D)^2$ where C and V are non-negative definite.

II. MAIN RESULTS

Definition 1. Let T ∈ B(H) be Drazin invertible, an operator T is called (N+K)-power -D-operator if $T^{*2}(T^D)^{2(n+k)}) = (T^*(T^D)^{n+k})^2$ for positive integers n and k.

Proposition 2. Let Tbe (n+k)-power D-operator, then the following holds;

i. λT∈ (n+k)-D-operator for every scalar λ.
ii. S ∈ (n+k)-D-operator for every S ∈ B(H) that is unitarily equivalent to T.
iii. The restriction/M of T to any closed subspace M of H which reduces T is in (n+k)-D-operator.
iv. $(T^D)^{n+k}$∈(n+k)-D-operator.

Proof.

i. The proof is trivial.
ii. Since S is unitarily equivalent to T, there exists a unitary operator U ∈ B(H) such that $S=UTU^*$. Hence;

$S^{*2n}(S^D)^{2(n+k)} = (UT^*U^*)^2(U(T^D)^{n+k}U^*)^2$

$= (UT^*U^*)(UT^*U^*)(U(T^D)^{n+k}U^*)(U(T^D)^{n+k}U^*)$
$=UT^*T^*(T^D)^{n+k}(T^D)^{n+k}U^*$
$= UT^{*2}(T^D)^{2(n+k)}U^*$
$=U(T^*(T^D)^{n+k})^2U^*$
$=UT^*(T^D)^{n+k}T^*(T^D)^{n+k}U^*$
$= (UT^*U^*)(U(T^D)^{n+k}U^*)(UT^*U^*)(U(T^D)^{n+k}U^*)$
$= S^*(S^D)^{n+k}S^*(S^D)^{n+k}$
$= (S^*(S^D)^{n+k})^{2.}$

Thus S ∈(n+k)-D-operator.

iii. $(T/M)^{*2}((T/M)D)^{2(n+k)}$
$=(T/M)^*(T/M)^*((T/M)^D)^{n+k}((T/M)^D)^{n+k}$
$= (T^*/M)(T^*/M)((T^D)^{n+k}/M)((T^D)^{n+k}/M)$
$= (T^*T^*/M)((T^D)^{n+k}(T^D)^{n+k}/M)^2$
$= (T^{*2}/M)((T^D)^{2(n+k)}/M)$
$= (T^{*2}(T^D)^{2(n+k)})/M$
$= (T^*(T^D)^{n+k}T^*(T^D)^{n+k})/M$
$= ((T^*(T^D)^{n+k})/M)((T^*(T^D)^{n+k})/M)$
$= ((T^*/M)((T^D)^{n+k}/M)(T^*/M)((T^D)^{n+k}/M))$
$= ((T^*/M)((T^D)^{n+k/}M)^2$
$= ((T/M)^*((T/M)^D)^{n+k})^{2.}$

Hence T/M ∈(n+k)-D-operator.

iv. Suppose T∈(n+k)-D-operator, then; $T^{*2n}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$, hence $T^*T^*(T^D)^{n+k}(T^D)^{n+k} = T^*(T^D)^{n+k}T^*(T^D)^{n+k}$ taking adjoins on both sides

$= ((T^*)^D)^{n+k} ((T^*)^D)^{n+k}TT = ((T^*)^D)^{n+k}T ((T^*)^D)^{n+k}T$. Thus $(((T^D)^{n+k})^*)^2 T^2 = (((T^D)^{n+k})^* T)^2$.

hence $(T^D)^{n+k} \in$(n+k)-D-operator.

Proposition 3. The set of all (n+k)-D-operator is a closed subset of B(H) on H.

Proof.
Let $\{T_q\}$ be a sequence of (n+k)-D-operator operators with $T_q \longrightarrow T$. We have to show that $T \in$(n+k)-D-operator. Now $T_q \to T$ implies $T_q^* \to T^*$ and $(T_q^D)^{n+k} \to (T^D)^{n+k}$ Thus $T_q^*(T_q^D)^{n+k} \to T^*(T^D)^{n+k}$ gives

$(T_q^*(T_q^D)^{n+k})^2 \to (T^*(T^D)^{n+k})^2$ …………. (0.1)

Similarly,
$T_q^{*2} \to T^{*2}$ and $(T_q^D)^{2(n+k)} \to (T^D)^{2(n+k)}$, thus

$T_q^{*2}(T_q^D)^{2(n+k)} \to T^{*2}(T^D)^{2(n+k)}$ ……………. (0.2)

hence from (0.1) and (0.2) we have;

$\| T^{*2}(T^D)^{2(n+k)} - (T^*(T^D)^{n+k})^2 \|$

$= \| T^{*2}(T^D)^{2(n+k)} - T_q^{*2}(T_q^D)^{2(n+k)} + T_q^{*2}(T_q^D)^{2(n+k)} - (T^*(T^D)^{n+k})^2 \|$

$\leq \| T^{*2}(T^D)^{2(n+k)} - T_q^{*2}(T_q^D)^{2(n+k)} \| + \| T_q^{*2}(T_q^D)^{2(n+k)} - (T^*(T^D)^{n+k})^2 \|$

$= \| T^{*2}(T^D)^{2(n+k)} - T_q^{*2}(T_q^D)^{2(n+k)} \| + \| T_q^{*2}((T_q^D)^{n+k})^2 - (T^*(T^D)^{n+k})^2 \| \to 0$ as $q \to \infty$ and thus

$T^{*2}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$ hence $T \in$(n+k)-D-operator.

- Proposition 4. Let $S, T \in$(n+k)-D-operator. If $[S, T] = [S, T^*] = 0$, then $TS \in$(n+k)-D-operator.

Proof
$[S, T] = [S, T^*] = 0$ implies;

$[S, T] = [S^D, T] = [S^*, T^D] = 0$ with $S, T \in$(n+k)-D-operatorwe have; $S^{*2}(S^D)^{2(n+k)} = (S^*(S^D)^{n+k})^2$ and

$T^{*2}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$, hence

$(TS)^{*2}((TS)^D)^{2(n+k)} = (TS)^*(TS)^*((TS)^D)^{n+k}((TS)^D)^{n+k}$
$= S^*T^*S^*T^*(T^D)^{n+k}(S^D)^{n+k}(T^D)^{n+k}(S^D)^{n+k}$
$= S^*S^*(S^D)^{n+k}(S^D)^{n+k}T^*T^*(T^D)^{n+k}(T^D)^{n+k}$
$= S^{*2}T^{*2}(S^D)^{2(n+k)}(T^D)^{2(n+k)}$
$= S^*S^*T^*T^*(S^D)^{n+k}(T^D)^{n+k}(S^D)^{n+k}(T^D)^{n+k}$
$= S^*T^*S^*T^*(S^D)^{n+k}(T^D)^{n+k}(S^D)^{n+k}(T^D)^{n+k}$
$= ((TS)^*(TS)^*((TS)^D)^{n+k})^2$.

Hence $TS \in$(n+k)-D-operator.

Proposition 5. Let $S, T \in$(n+k)-D-operator.If $TS = ST = 0$, then $S+T \in$(n+k)-D-operator.

Proof.
$S, T \in$(n+k)-D-operator implies; $S^{*2}(S^D)^{2(n+k)} = (S^*(S^D)^{n+k})^2$ and
$T^{*2}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$.
$TS=ST=0$ implies $T^*S^* = S^*T^*$ which further implies $((S + T)^D)^{n+k} = (SD)^{n+k} + (T^D)^{n+k}$. Thus,
$= (S + T)^{*2}((S + T) D)^{2(n+k)} = (S + T)^*(S + T)^*((S + T) D)^{n+k}((S + T) D)^{n+k}$

$= (S^* + T^*)(S^* + T^*)(S^D + T^D)^{n+k}(S^D + T^D)^{n+k}$
$= (S^{*2} + T^{*2})((S^D)^{2(n+k)} + (T^D)^{2(n+k)})$
$= S^{*2}(S^D)^{2(n+k)} + T^{*2}(T^D)^{2(n+k)}$
$= (S^*(S^D)^{n+k})^2 + (T^*(T^D)^{n+k})^2$
$= (S^*(S^D)^{n+k} + T^*(T^D)^{n+k})(S^*(S^D)^{n+k} + T^*(T^D)^{n+k})$
$= (S^* + T^*)((S^D)^{n+k} + (T^D)^{n+k})(S^* + T^*)((S^D)^{n+k} + (T^D)^{n+k})$
$= ((S + T)^*((S + T)^D)^{n+k})^2$.

Hence $S+T \in$(n+k)-D-operator.

Theorem 6. Let $T_{\alpha 1}, T_{\alpha 2}.........T_{\alpha q} \in$(n+k)-D-operator. , then it follows that;

i. $T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha q} \in$(n+k)-D-operator.
ii. $T_{\alpha 1} \otimes T_{\alpha 2} \otimes ...... \otimes T_{\alpha q} \in$(n+k)-D-operator.

Proof. (i) . $T_{\alpha j} \in$ (n+k)-D-operator for all $\alpha j = 1, 2, ...\alpha q$ implies;
$T_{\alpha j}^{*2}(T_{\alpha j}^D)^{2(n+k)} = (T_{\alpha j} * (T_{\alpha j}^D)^{n+k})^2$ thus
$(T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha j})^{*2}((T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha j})^D)^{2(n+k)}$
$= T_{\alpha 1}^{*2}(T_{\alpha 1}^D)^{2(n+k)} \oplus T_{\alpha 2}^{*2}(T_{\alpha 2}^D)^{2(n+k)} \oplus ...... \oplus T_{\alpha j}^{*2}(T_{\alpha j}^D)^{2(n+k)}$
$= (T_{\alpha 1}^*(T_{\alpha 1}^D)^{n+k})^2 \oplus (T_{\alpha 2}^*(T_{\alpha 2}^D)^{n+k})^2 \oplus ...... \oplus (T_{\alpha j}^*(T_{\alpha}$

$_j{}^D)^{n+k})^2$
$= T_{\alpha 1^*} (T_{\alpha 1}{}^D)^{n+k} T_{\alpha 1^*} (T_{\alpha 1}{}^D)^{n+k} \oplus T_{\alpha 2^*} (T_{\alpha 2}{}^D)^{n+k} T_{\alpha 2^*} (T_{\alpha 2}{}^D)^{n+k} \oplus \ldots\ldots \oplus T_{\alpha j^*} (T_{\alpha j}{}^D)^{n+k} T_{\alpha j^*} (T_{\alpha j}{}^D)^{n+k}$
$= T_{\alpha 1^*} (T_{\alpha 1}{}^D)^{n+k} \oplus T_{\alpha 2^*} (T_{\alpha 2}{}^D)^{n+k} \oplus \ldots\ldots \oplus T_{\alpha j^*} (T_{\alpha j}{}^D)^{n+k}$
$= ((T_{\alpha 1^*} \oplus T_{\alpha 2^*} \oplus \ldots\ldots \oplus T_{\alpha j^*}) ((T_{\alpha 1}{}^D)^{n+k} \oplus (T_{\alpha 2}{}^D)^{n+k} \oplus \ldots\ldots \oplus (T_{\alpha j}{}^D)^{n+k}))$
$= ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^* ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^D)^{n+k})^2$

iii. The proof for (ii) follows similarly.

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