

On Extension Of D-Operator Operators

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Abstract- In this paper, we generalize the class of D-Operator by extending this study to n-D-Operator and investigate the basic properties of this class. We also show that this class is closed under strong operator topology.

Indexed Terms- D-Operator, Class (Q), Almost Class (Q), (α, β)-Class (Q), Normal operators, n-Normal, n-D-Operator operators.

I. INTRODUCTION

Throughout this paper, H is a separable complex Hilbert space, B(H) is the Banach algebra of all bounded linear operators. n-normal if $T^* T^n = T^n T^*$, $T \in B(H)$ is normal if $T^* T = T T^*$, quasinormal if $T (T^* T) = (T^* T) T$. D-Operator if $T^{*2} (T^D)^2 = (T^* T^D)^2$ (1), class (Q) if $T^{*2} T^2 = (T^* T)^2$ (5), n-power class (Q) if $T^{*2} (T^n)^2 = (T^* T^n)^2$ (6), n-D-Operator if $T^{*2} (T^D)^{2n} = (T^* (T^D)^n)^2$, for any positive integer n. We note that n-D-Operator is D-Operator when n=1.

II. MAIN RESULTS

Definition 1. Let $T \in B(H)$ be Drazin invertible. Then an operator T is called nD-Operator, denoted by, [nD], if $T^{*2} (T^D)^{2n} = (T^* (T^D)^n)^2$, for any positive integer n.

Proposition 2. Let $T \in [nD]$, then the following holds;

- i. $\lambda T \in [nD]$ for every scalar λ .
- ii. $S \in [nD]$ for every $S \in B(H)$ that is unitarily equivalent to T.
- iii. The restriction/M of T to any closed subspace M of H which reduces T is in [nD].
- iv. $(T^D)^n \in [nD]$.

Proof.

- i. The proof is trivial.
- ii. Since S is unitarily equivalent to T, there exists a unitary operator $U \in B(H)$ such that $S=UTU^*$. Hence;

$$\begin{aligned} S^{*2n} (S^D)^{2n} &= (UT^*U^*)^2 (U (T^D)^n U^*)^2 \\ &= (UT^*U^*) (UT^*U^*) (U (T^D)^n U^*) (U (T^D)^n U^*) \\ &= UT^*T^* (T^D)^n (T^D)^n U^* \\ &= UT^{*2} (T^D)^{2n} U^* \\ &= U (T^* (T^D)^n)^2 U^* \\ &= UT^* (T^D)^n T^* (T^D)^n U^* \\ &= (UT^*U^*) (U (T^D)^n U^*) (UT^*U^*) (U (T^D)^n U^*) \\ &= S^*(S^D)^n S^*(S^D)^n \\ &= (S^*(S^D)^n)^2. \end{aligned}$$

Thus $S \in [nD]$.

$$\begin{aligned} \text{iii. } (T/M)^{*2} ((T/M)^D)^{2n} &= (T/M)^* (T/M)^* ((T/M)^D)^{2n} \\ &= (T^*/M) (T^*/M) ((T^D)^n/M) ((T^D)^n/M) \\ &= (T^* T^*/M) ((T^D)^n/M)^2 \\ &= (T^{*2}/M) ((T^D)^{2n}/M) \\ &= (T^{*2} (T^D)^{2n})/M \\ &= (T^* (T^D)^n T^* (T^D)^n)/M \\ &= ((T^* (T^D)^n)/M) ((T^* (T^D)^n)/M) \\ &= ((T^*/M) ((T^D)^n/M) (T^*/M) ((T^D)^n/M)) \\ &= ((T^*/M) ((T^D)^n/M))^2 \\ &= ((T/M)^* ((T/M)^D)^n)^2. \end{aligned}$$

Hence $T/M \in [nD]$.

iv. Suppose $T \in [nD]$, then;

$$\begin{aligned} T^{*2n} (T^D)^{2n} &= (T^* (T^D)^n)^2, \text{ hence} \\ T^* T^* (T^D)^n (T^D)^n &= T^* (T^D)^n T^* (T^D)^n \\ \text{taking adjoints on both sides} \\ &= ((T^*)^D)^n ((T^*)^D)^n T T = ((T^*)^D)^n T ((T^*)^D)^n T. \\ \text{Thus } (((T^D)^n)^*)^2 T^2 &= (((T^D)^n)^*)^2 T^2. \end{aligned}$$

hence $(T^D)^n \in [nD]$.

Proposition 3. The set of all n-D-Operators is a closed subset of B(H) on H.

Proof.

Let $\{T_q\}$ be a sequence of [nD] operators with $T_q \rightarrow T$. We have to show that $T \in [nD]$. Now $T_q \rightarrow T$ implies $T_q^* \rightarrow T^*$ and $(T_q^D)^n \rightarrow$

$(T^D)^n$. Thus $T_q * (T_q^D)^n \rightarrow T * (T^D)^n$ gives
 $(T_q * (T_q^D)^n)^2 \rightarrow (T * (T^D)^n)^2 \dots\dots\dots (0.1)$

Similarly,
 $T_q^{*2} \rightarrow T^{*2}$ and $(T_q^D)^{2n} \rightarrow (T^D)^{2n}$, thus

$$T_q^{*2} (T_q^D)^{2n} \rightarrow T^{*2} (T^D)^{2n} \dots\dots\dots (0.2)$$

hence from (0.1) and (0.2) we have;

$$\begin{aligned} & \left\| T^{*2} (T^D)^{2n} - (T * (T^D)^n)^2 \right\| \\ &= \left\| T^{*2} (T^D)^{2n} - T_q^{*2} (T_q^D)^{2n} + T_q^{*2} (T_q^D)^{2n} - (T * (T^D)^n)^2 \right\| \\ &\leq \left\| T^{*2} (T^D)^{2n} - T_q^{*2} (T_q^D)^{2n} \right\| + \left\| T_q^{*2} (T_q^D)^{2n} - (T * (T^D)^n)^2 \right\| \\ &= \left\| T^{*2} (T^D)^{2n} - T_q^{*2} (T_q^D)^{2n} \right\| + \left\| T_q^{*2} ((T_q^D)^n)^2 - (T * (T^D)^n)^2 \right\| \rightarrow 0 \text{ as } q \rightarrow \infty \text{ and thus} \end{aligned}$$

$T^{*2} (T^D)^{2n} = (T * (T^D)^n)^2$ hence $T \in [nD]$.

Proposition 4. Let $S, T \in [nD]$. If $[S, T] = [S, T^*] = 0$, then $TS \in [nD]$.

Proof
 $[S, T] = [S, T^*] = 0$ implies;

$[S, T] = [S^D, T] = [S^*, T^D] = 0$ with $S, T \in [nD]$ we have; $S^{*2} (S^D)^{2n} = (S^* (S^D)^n)^2$ and $T^{*2} (T^D)^{2n} = (T * (T^D)^n)^2$, hence

$$\begin{aligned} (TS)^{*2} ((TS)^D)^{2n} &= (TS)^* (TS)^* (TS)^D (TS)^D \\ &= S^* T^* S^* T^* (T^D)^n (S^D)^n (T^D)^n (S^D)^n \\ &= S^* S^* (S^D)^n (S^D)^n T^* T^* (T^D)^n (T^D)^n \\ &= S^{*2} T^{*2} (S^D)^{2n} (T^D)^{2n} \\ &= S^* S^* T^* T^* (S^D)^n (T^D)^n (S^D)^n (T^D)^n \\ &= S^* T^* S^* T^* (S^D)^n (T^D)^n (S^D)^n (T^D)^n \\ &= (TS)^* (TS)^* ((TS)^D)^{2n}. \end{aligned}$$

Hence $TS \in [nD]$.

Proposition 5. Let $S, T \in [nD]$. If $TS = ST = 0$, then $S+T \in [nD]$.

Proof.
 $S, T \in [nD]$ implies; $S^{*2} (S^D)^{2n} = (S^* (S^D)^n)^2$ and

$$\begin{aligned} T^{*2} (T^D)^{2n} &= (T * (T^D)^n)^2. \\ TS = ST = 0 &\text{ implies } T^* S^* = S^* T^* \text{ which further implies} \\ &((S + T)^D)^n = (S^D)^n + (T^D)^n. \text{ Thus,} \\ &= (S + T)^{*2} ((S + T)^D)^{2n} = (S + T)^* (S + T)^* ((S + T)^D)^n ((S + T)^D)^n \\ &= (S^* + T^*) (S^* + T^*) (S^D + T^D)^n (S^D + T^D)^n \\ &= (S^{*2} + T^{*2}) ((S^D)^{2n} + (T^D)^{2n}) \\ &= S^{*2} (S^D)^{2n} + T^{*2} (T^D)^{2n} \\ &= (S^* (S^D)^n)^2 + (T^* (T^D)^n)^2 \\ &= (S^* (S^D)^n + T^* (T^D)^n) (S^* (S^D)^n + T^* (T^D)^n) \\ &= (S^* + T^*) ((S^D)^n + (T^D)^n) (S^* + T^*) ((S^D)^n + (T^D)^n) \\ &= ((S + T)^* ((S + T)^D)^n)^2. \end{aligned}$$

Hence $S+T \in [nD]$.

Theorem 6. Let $T_{\alpha 1}, T_{\alpha 2}, \dots, T_{\alpha q} \in [nD]$, then it follows that;

- i. $T_{\alpha 1} \oplus T_{\alpha 2} \oplus \dots \oplus T_{\alpha q} \in [nD]$.
- ii. $T_{\alpha 1} \otimes T_{\alpha 2} \otimes \dots \otimes T_{\alpha q} \in [nD]$.

Proof.

$T_{\alpha j} \in [nD]$ for all $\alpha j = 1, 2, \dots, \alpha q$ implies;
 $T_{\alpha j}^{*2} (T_{\alpha j}^D)^{2n} = (T_{\alpha j} * (T_{\alpha j}^D)^n)^2$ thus
 $(T_{\alpha 1} \oplus T_{\alpha 2} \oplus \dots \oplus T_{\alpha q})^{*2} ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \dots \oplus T_{\alpha q})^D)^{2n}$
 $= T_{\alpha 1}^{*2} (T_{\alpha 1}^D)^{2n} \oplus T_{\alpha 2}^{*2} (T_{\alpha 2}^D)^{2n} \oplus \dots \oplus T_{\alpha j}^{*2} (T_{\alpha j}^D)^{2n}$
 $= (T_{\alpha 1} * (T_{\alpha 1}^D)^n)^2 \oplus (T_{\alpha 2} * (T_{\alpha 2}^D)^n)^2 \oplus \dots \oplus (T_{\alpha j} * (T_{\alpha j}^D)^n)^2$
 $= T_{\alpha 1}^* (T_{\alpha 1}^D)^n T_{\alpha 1} * (T_{\alpha 1}^D)^n \oplus T_{\alpha 2}^* (T_{\alpha 2}^D)^n T_{\alpha 2} * (T_{\alpha 2}^D)^n \oplus \dots \oplus T_{\alpha j}^* (T_{\alpha j}^D)^n T_{\alpha j} * (T_{\alpha j}^D)^n$
 $= T_{\alpha 1}^* (T_{\alpha 1}^D)^n \oplus T_{\alpha 2}^* (T_{\alpha 2}^D)^n \oplus \dots \oplus T_{\alpha j}^* (T_{\alpha j}^D)^n$
 $= ((T_{\alpha 1}^* \oplus T_{\alpha 2}^* \oplus \dots \oplus T_{\alpha j}^*) ((T_{\alpha 1}^D)^n \oplus (T_{\alpha 2}^D)^n \oplus \dots \oplus (T_{\alpha j}^D)^n))$
 $= ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \dots \oplus T_{\alpha j})^* ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \dots \oplus T_{\alpha j})^D)^n)^2$

(ii) . The proof for (ii) follows similarly.

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