On N Quasi (m, p+k)-Power D-Operator Operators

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Abstract- In this paper, we introduce the class of (p+k)-D-Operatoracting on the usual Hilbert space H over the complex plane. An operator T is said to be an (p+k)-D-Operatorif T $(T^{*2}(T^{D})^{2(p+k)}) = N(T^{*}(T^{D})^{p+k})^2T$ for positive integers p and k and for N which is a bounded operator on H. We investigate thebasic behavior of this class of operator.

Indexed Terms- Normal operators, D-Operator, Almost Class (Q), quasi -class (Q) operators, N quasi-D-operator.

I. INTRODUCTION

H denotes the separable complex Hilbert space in this paper, while B(H) is the usual Banach algebra of all bounded linear operators on H. Let T ∈ B(H), Drazin inverse of T is an operator $T^{D} \in B(H)$, such that $TT^{D} = T^{D}T$, $T^{D} = T^{D}TT^{D}$ and $T^{k+1}T^{D} = T^{k}$ provided it exists. An operator T ∈ B(H) is said to be D-Operator if $T^{*2}(T^{D})^2 = (T^{*}T^{D})^2$ (1), class (Q) if $T^{*2}T^{2} = (T^{*}T)^2$ (4), M Quasi class (Q) if $T(T^{*2}T^{2}) = M(T^{*}T)^2T$ (5), Quasi class (Q) if $T(T^{*2}T^{2}) = (T^{*}T)^2T$, N quasi-D-Operator if $T(T^{*2}(T^{D})^2) = N(T^{*}T^{D})^2T$, for a bounded linear operator N. Let T = ξ + iζ, with ξ = Re(T) $=\frac{TD+T*}{2}$ and ζ = Im(T) = $\frac{TD-T*}{2i}$. We shall simply denote $U^{2} = (T^{*}T^{D})^2$ and $V^{2} = T^{*2}(T^{D})^2$ where C and V are non-negative definite.

II. MAIN RESULTS

- Definition 1. Let T ∈ B(H) be Drazin invertible, an operator T is called (m, p+k)-D-Operatorif $T(T^{*2m}(T^{D})^{2(p+k)}) = N(T^{*m}(T^{D})^{p+k})^2T$ for positive integers p and k and N which is a bounded operator on H.

- Theorem 2. Let T ∈ B(H) and let V commute with ξ and ζ such that $V^{2}T = NU^{2}T$, it follows that T is an (m, p+k)-D-Operator.

Proof. We recall that T = ξ + iζ, with ξ = Re(T) = $\frac{TD+T*}{2}$ and ζ = Im(T) = $\frac{TD-T*}{2i}$ and $U^{2} = (T^{*m}(T^{D})^{p+k})^2$ and $V^{2} = T^{*2m}(T^{D})^{2(p+k)}$. Since V ξ = ξV and Uζ = ζU, we have;

$V^{2}\xi = \xi V^{2}$ and $U^{2}\zeta = \zeta U^{2}$, thus
$V^{2}T + V^{2}(T)^{*} = TV^{2} + (T)^{*}V^{2}$
$V^{2}T - V^{2}(T)^{*} = TV^{2} - (T)^{*}V^{2}$ implies;
$TV^{2} = V^{2}T$. Hence;
$T(T^{*2m}(T^{D})^{2(p+k)}) = ((T^{*m}(T^{*m}(T^{D})^{p+k})(T^{D})^{p+k})$
$T = (T^{*m}(T^{D})^{p+k})^2T$.
$TU^{2} = NU^{2}T$ implies;
$T(T^{*2m}(T^{D})^{2(p+k)}) = N((T^{*m}(T^{*m}(T^{D})^{p+k})(T^{D})^{p+k})T$
$T(T^{*2m}(T^{D})^{2(p+k)}) = N(T^{*m}(T^{D})^{p+k})^2T$

Hence T is an (m, p+k)-D-Operator.

Proposition 3. Let T ∈ B(H) be a (m, p+k)-D-operator where $V^{2}\xi = \frac{1}{N}\xi V^{2}$ and $V^{2}\zeta = \frac{1}{N}\zeta V^{2}$, then T is an (m, p+k)-D-Operator.

Proof. $V^{2}\xi = \frac{1}{N}\xi V^{2}$ and $V^{2}\zeta = \frac{1}{N}\zeta V^{2}$ implies
$V^{2}(\xi + i\zeta) = \frac{1}{N}(\xi + i\zeta)V^{2}$
$V^{2}T = \frac{1}{N}TV^{2}$
$(T^{*m}(T^{*m}(T^{D})^{p+k})(T^{D})^{p+k})T = \frac{1}{N}T(T^{*m}(T^{*m}(T^{D})^{p+k})(T^{D})^{p+k})$
$T(T^{*m}(T^{*m}(T^{D})^{p+k})(T^{D})^{p+k}) = N(T^{*m}(T^{*m}(T^{D})^{p+k})(T^{D})^{p+k})T$
$= N(T^{*m}(T^{D})^{p+k})^2$ (Since T is a (m,p+k)- D-operator).

Hence T is an (m,p+k)-D-Operator.

- Theorem 4. Let T_α and T_β be two N Quasi-(m,p+k)-D-Operators from B (H, H) such that $(T_\alpha^{D})^{p+k}T_\beta^{*2m} = (T_\beta^{D})^{p+k}T_\alpha^{*2m} = T_\alpha^{*2m}(T_\beta^{D})^{2(p+k)} = T_\beta^{*2m}(T_\alpha^{D})^{2(p+k)} = 0$, then $T_\alpha + T_\beta$ is an N Quasi-(p+k)-D-Operator.

Proof. Since T_α and T_β are N Quasi-(p+k)-D-Operator, we have;
$(T_\alpha + T_\beta)[(T_\alpha + T_\beta)^{*2m}(T_\alpha^{D} + T_\beta^{D})^{2(p+k)}] = (T_\alpha + T_\beta)[(T_\alpha^{*2m} + T_\beta^{*2m})((T_\alpha^{D})^{2(p+k)} + (T_\beta^{D})^{2(p+k)})$
$=(T_\alpha + T_\beta)[T_\beta^{*2m}(T_\alpha^{D})^{2(p+k)} + T_\beta^{*2m}(T_\beta^{D})^{2(p+k)} +$

$T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)} + T_\alpha{}^{*2m}(T_\beta{}^D)^{2(p+k)}]$
$= (T_\alpha + T_\beta)[T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)} + T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)}]$ since
$T_\beta{}^{*2m}(T_\alpha{}^D)^{2(p+k)} = T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)} = 0$
$= (T_\alpha + T_\beta)[T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)} + T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)}]$
$= T_\alpha T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)} + T_\beta T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)}$ since
$T_\alpha T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)} = T_\beta T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)} = 0$
$= N(T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)})T_\alpha + N(T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)})T_\beta$
$= N(T\alpha^{*m}(T_\alpha{}^D)^{p+k})^2 T_\alpha + N(T_\beta{}^{*m}(T_\beta{}^D)^{p+k})^2 T_\beta$

Thus $T_\alpha + T_\beta$ is an (m, p+k)-D-Operator.

- Theorem 5. Let T_α and T_β be two N Quasi- (p+k)-D-Operatorfrom B (H, H) such that $(T_\alpha{}^D)^{p+k}$

$T_\beta{}^{*2m} = (T_\beta{}^D)^{p+k} T_\alpha{}^{*2m} = T_\alpha{}^{*2m}(T_\beta{}^D)^{2(p+k)} =$ $T_\beta{}^{*2m}(T_\alpha{}^D)^{2(p+k)} = 0$, then T_α - T_β is an N Quasi- (m, p+k)-D-Operator.

Proof. The proof follows from Theorem 4 above.

- Theorem 6. Let T_α and T_β be two N Quasi -(p+k)-D-Operators, then T_α T_β is an N Quasi -(p+k)-D-Operator provided $T_\alpha T_\beta = T_\beta T_\alpha$ and $(T_\alpha{}^D)^{2(p+k)} T_\beta{}^{*2m} = T_\beta{}^{*2m}(T_\alpha{}^D)^{2(p+k)}$.

Proof. Since T_α and T_β are N Quasi-(p+k)-D-Operator, we have ;
$(T_\alpha T_\beta)[(T_\alpha T_\beta)^{*2m}((T_\alpha T_\beta)^D)^{2(p+k)}]$
$=(T_\alpha T_\beta)[(T_\alpha{}^{*2m} T_\beta{}^{*2m})(T_\alpha{}^D T_\beta{}^D)^{2(p+k)}]$
$= (T_\alpha T_\beta)[(T_\beta{}^{*2m} T\alpha^{*2m})(T_\alpha{}^D T_\beta{}^D)^{2(p+k)}]$
$= T_\alpha(T_\beta T_\alpha{}^{*2m})(T_\beta{}^{*2m}(T_\alpha{}^D)^{2(p+k)})(T_\beta{}^D)^{2(p+k)}$
$= T_\alpha(T_\alpha{}^{*2m} T_\beta)(T_\beta{}^{*2m}(T_\alpha{}^D)^{2(p+k)})(T_\beta{}^D)^{2(p+k)}$
$= T_\alpha T_\alpha{}^{*2m} T_\beta(T_\alpha{}^D)^{2(p+k)} T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)}$
$= T_\alpha T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)} T_\beta T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)}$
$= N(T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)})T_\alpha N(T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)})T_\beta$
$= N(T_\alpha{}^{*2m}((T_\alpha{}^D)^{2(p+k)} T_\alpha)(T_\beta{}^{*2m}(T_\beta{}^D)^{2(p+k)}))T_\beta)$
$=N(T_\alpha{}^{*2m}(T_\alpha{}^D)^{2(p+k)} T_\beta{}^{*2m} T_\alpha(T_\beta{}^D)^{2(p+k)} T_\beta)$
$=N(T_\alpha{}^{*2m} T_\beta{}^{*2m}(T_\alpha{}^D)^{2(p+k)}(T_\beta{}^D)^{2(p+k)} T_\alpha T_\beta)$
$=N[(T_\alpha T_\beta)^{*2m}(T_\alpha{}^D T_\beta{}^D)^{2(p+k)}(T_\alpha T_\beta)]$
$=N[(T_\alpha T_\beta)^{*2m}((T_\alpha T_\beta)^D))^{2(p+k)}(T_\alpha T_\beta)]$
$=N[(T_\alpha T_\beta)^{*m}((T_\alpha T_\beta)^D)^{p+k}]^2(T_\alpha T_\beta)$

Thus T_α T_β is N Quasi -(m, p+k)-D-Operator.

- Theorem 7. Power of N Quasi D-operator is similarly N Quasi- (m, p+k)-D-Operator.

Proof. We first show that the result holds for some p =1, then we have;
$T(T^{*2m}(T^D)^{2(p+k)}) = N(T^{*m}(T^D)^{p+k})^2 T$………………………. (0.1)

Suppose the result holds for p=n, we have;
$[T(T^{*2m}(T^D)^{2(p+k)})]^n = (N(T^{*m}(T^D)^{p+k})^2 T)^n$…………………. (0.2)

we then prove that the result is true for p=n+1. We have;
$[T(T^{*2m}(T^D)^{2(p+k)})]^{n+1} = (N(T^{*m}(T^D)^{p+k})^2 T)^{n+1}$………………… (0.3)
$[T(T^{*2m}(T^D)^{2(p+k)})]^{n+1} = [NT(T^{*2m}(T^D)^{2(p+k)})]^n [NT(T^{*2m}(T^D)^{2(p+k)})]$ ………. (0.4)
$= [N(T^{*m}(T^D)^{p+k})^2 T]^n [N(T^{*m}(T^D)^{p+k})^2 T]$ by (0.1) and (0.2)
$[T(T^{*2m}(T^D)^{2(p+k)})]^{n+1} = [N(T^{*m}(T^D)^{p+k})^2 T]^{n+1}$………………………. (0.5)

Hence the proof as required.

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