A Review of Rules of Differentiation by Differentiation from First Principles

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Abstract- In this paper, nine basic rules of differentiation are reviewed by the basic principle usually known as differentiation from first principles. It entails deriving the constant, constant multiple, sum, difference, product, quotient, reciprocal, power and chain rules, which are then used to find derivatives of selected functions. Some fundamental concepts describing the basic principle are reviewed. Particularly, the concepts limit, continuity of a function, differentiability of a function and derivative of a function are thoroughly reviewed. Analytically, the rules are obtained; and it is established that they usually take less amount of computation when the derivative of a function is calculated. Thus, the rules are easier to manipulate than the basic method. It is recommended that calculus learners should employ both methods when computing the derivative of a function to acquire full understanding of the concept differentiation. They should know exactly that the rules originated from the fundamental principle; and with them computations are easier.

Indexed Terms- Function, Limit of a Function, Derivative of a Function, Differentiation from First Principles, Rules of Differentiation.

I. INTRODUCTION

A. Background of the Study

Classical calculus, in modern form, was developed in the end of seventeenth century by Isaac by well-known mathematicians Isaac Newton (1642–1727) and Gottfried Leibniz (1646-1716) [12]. It is a mathematical discipline that is largely concerned with functions, limits, derivatives and integrals. They worked independently to develop these essential concepts that were later made accurate by other mathematicians [15]. While it is split between the two definitions of Newton and Leibniz, it has still been able to produce a new mathematical system and is used in a variety of applications.

There are two different branches of calculus, namely differential calculus and integral calculus.

The differential calculus is one of the materials in mathematics that addresses determination of a derivative function. In determining the derivative of a function, the function is required to be differentiable in an open interval. The derivative process is called differentiation whereby the derivative is intimately related to the differential. The differential portrays changes in a function with a small change in the function variable. Moreover, it discusses solving problems related to changes in variables that form a mathematical model. Mathematical models that are connected to differential can be applied in a variety of fields such as physics, biology, engineering, medicine and economics.

This review work aims at showing overtly the origin of differentiation as a fundamental process of finding the derivative of a function $y = f(x)$ at a generic point $P(x, y)$. This is preceded by a review on the concepts limit, continuity, derivative and differentiability of the function; and how they are related. In particular, differentiation is a general process that is used to find the slope of a line or curve. But, for a curve, the slope is always varying. Thus, the gradient of the curve at a given point is defined as the slope of the tangent at that point. The function that describes the slope of the tangent at a given point is called the derivative [14]. Algebraically, the gradient or slope of the function $f(x)$ at the point $P(x, y)$ is expressed as

$$f'(x) = \lim_{\Delta X \to 0}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right),$$

where Δx is a small increment in the dependent variable x. This limit is also the instantaneous rate of change of y with respect x; and the process is known as differentiation from first principles [2].

In essence, the review of major differential calculus concepts and main results of the study are aimed at helping instructors to apply appropriate pedagogical methods and techniques to tackle calculus learners' misconceptions that commonly happen in colleges, universities and other education institutions. In these educational settings, some learners believe that the limit and the function are the same [1], [5], [7], [10]; and others think that if the function is defined at a point, it is also continuous at the point [5] which is untrue. Besides, the rules for differentiation are often introduced as a list leading to rote-learning [9], which is in contrast to good instructive practice [8]. However, the study mostly shows how the rules are obtained from the fundamental principle introduced by Leibniz and Newton. Actually, these rules serve as 'shortcuts' when computing the derivative of a function, where a lesser amount of computation is observed.

B. Preliminaries

This section presents a review on some mathematical aspects instituting the differential calculus (DC). At this point, the concepts of limit, continuity, derivative and differentiability of a function are systematically discussed. Besides, the fundamental principle of differentiation is reviewed. With these fundamental pieces of the DC, the rules under consideration can easily be presented and discussed.

The Limiting Value of a Function

Definition 1: *Let* $f(x)$ *be defined for all* x *in an interval about* $x=a$. *Then the function* $f(x)$ *has a limit* L *at* $x=a$ *if for all* $\varepsilon>0$ *there exists a* $\delta>0$ *such that* $0<\left|f(x)-L\right|<\delta$ *implies that* $\left|f(x)-L\right|<\varepsilon$

This definition was developed around 1800" s by famous mathematicians Karl Theodor Wilhelm Weierstrass (1815-1897), Bernard Placidus Johan Nepomuk Bolzano (1781-1848) and Augustin Louis Cauchy (1789-1857) [6].

In this case, it is realized the variable x has values lying in an interval characterized by

i. The interval is centered at some number a
ii. The difference between x and a must be less than another positive number δ.
iii. The variable x cannot have the particular value.

This can be summarized in the following way:

$\left|x-a\right|>0$ and $\left|x-a\right|<\delta$

These relations can be combined in a single statement as

$$0<\left|x-a\right|<\delta$$

This means that the difference between $f(x)$ and L is less than another positive number ε, which corresponds to $\delta>0$. The number δ usually depends on how the number ε is selected. Undoubtedly, the limit tells where the value of a function approaches as the function inputs gets closer and closer to some number. If this can always be executed then the argument holds and the limit L exits.

Properties of Limits

Suppose $f(x)$ and $g(x)$ are functions defined on an interval (α,β) where $a\in(\alpha,\beta)$. If the limits $\lim_{x\to a} f(x)=A$ $\lim_{x\to a} g(x)=B$ both exist and are finite, then

i. $\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$
$=A+B$
ii. $\lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$
$=A-B$
iii. $\lim_{x\to a}\left(f(x)\times g(x)\right)=\lim_{x\to a} f(x)\times\lim_{x\to a} g(x)$
$=A\times B=AB$
iv. $\lim_{x\to a}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\dfrac{A}{B}$ provided that $A\neq B$
v. $\lim_{x\to a} kf(x)=k\lim_{x\to a}=kA$
vi. $\lim_{x\to a}\left(f(x)\right)^p=\left(\lim_{x\to a} f(x)\right)^p=A^p$

vii. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A}$

Obviously, these limit properties are tremendously useful when evaluating a limit. These results were obtained following comprehensive work pertaining to evaluation of limits.

The Continuity of a Function

Definition 2: *Let $f(x)$ be a function and let a be in the domain of $f(x)$. Then the function is continuous at $x = a$ if and only if for $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $0 < |x - a| < \delta$*

This definition is similar to the definition of a limit but the only distinction between them is that L is replaced by $f(a)$. With regards to this, it is deduced that $\lim_{x=a} f(x) = f(a)$.

Thus, if a function f is continuous at every point where it is defined, then it is a continuous function.

The Derivative of a Function

Definition 3: *Let a function $f(x)$ be continuous at x. Then the derivative of $f(x)$ with respect to x* is the function $f'(x)$ defined as

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

This is the result based on the following argument:

Suppose $y = f(x)$ is continuous function on the interval $[\alpha, \beta]$ and $(x_0, f(x_0))$ is a point lying on the graph of $y = f(x)$ for all $x_0 \in [\alpha, \beta]$. Let Δx be an small increment in x, and let Δy be an increment in y corresponding to Δx. Then any change in the variable x from x_0 to $x_0 + \Delta x$ makes y change from y_0 to $y_0 + \Delta y$ such that points $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$ all lie on the graph of $y = f(x)$. With regards to this, we have

$$y_0 + \Delta y = f(x_0 + \Delta x)$$

$$\Delta y = f(x_0 + \Delta x) - y_0$$

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \qquad (y_0 = f(x_0))$$

Hence the ratio $\frac{\Delta y}{\Delta x}$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The slope of the function $f(x)$ at x_0 is given by the limit of $\Delta y / \Delta x$ as Δx approaches zero. If the limit exists then it is called the derivative of the function $f(x)$ at $x = x_0$

This limit is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Hence the derivative of the function $f(x)$ at x is given by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

This derivative can also be expressed as

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

[11] define the derivative of a function $f(x)$ at a point a as the limit of the difference quotients (if it exists):

$$f'(a) = \lim_{\Delta x \to 0}\left[\frac{f(a+\Delta x) - f(a)}{\Delta x}\right]$$

Equivalently, the derivative of the function $f(x)$ at a point a is

$$f'(a) = \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a}\right]$$

With regards to the definition of derivative, it implies that the limit exits when a function is differentiable at a given point. This establishes Theorem 1.

Theorem 1: *A function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists.*

Theorem 2: *A differentiable function is continuous*.

This means if $f(x)$ is differentiable at $x = a$ it is also continuous at $x = a$ [3]

Proof: Since the function $f(x)$ is differentiable at $x=a$, then there exists $f'(x)$ such that

$$f'(x)=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right]$$

Then

$$\lim_{x\to a}\left(f(x)-f(a)\right)=\lim_{x\to a}\left[(x-a)\frac{f(x)-f(a)}{(x-a)}\right]$$

This holds only if $x-a\neq 0$ for a limit at $x=a$

Thus, $f'(x)=0$. Hence, $\lim_{x\to a}\left(f(x)-f(a)\right)=0$.

Adding $f(a)$ on both sides gives

$$\lim_{x=a} f(x)=f(a)$$

This is the definition of continuity of $f(x)$ at $x=a$.

Other Notations for the First Derivative

In practice, there are many notations that connote the derivative of the function $y=f(x)$. Some of the notations are

$$\frac{dy}{dx}=\frac{d}{dx}y=y'=f'(x)=\frac{d}{dx}f(x)=\frac{df}{dx}$$

Others are

$$Df(x)=D_x f(x)=\dot{y}$$

where $\frac{d}{dx}$, $D=\frac{d}{dx}$ and $D_x=\frac{d}{dx}$ are called differentiation operators. The differentiation operator is the derivative operator that operates on a function to produce Δx and Δy and evaluates $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ to produce the derivative function. Gottfried Leibnitz introduced the notation $\frac{dy}{dx}$ for the derivative of $f(x)$ and used the symbol $\frac{d^n y}{dx^n}$ to represent the n-th-order derivative [4] ; Joseph- Luis Lagrange, the prime notation $f'(x)$ to connote the derivative of the function $f(x)$; Sir Isaac Newton, the dot notation $\dot{y}$, $\ddot{y}$,... to mean the first, second and higher derivatives and Leonhard Euler, the operator notation $D_x y$ to represent the derivative of $y=f(x)$ with respect to x. Overall, the derivative is a measure of instantaneous rate of change of $y=f(x)$ with respect to a change in x.

II. METHODOLOGY

This study is primarily a review on the rules of differentiation. All information presented in the review largely depends on secondary data obtained from different sources: books, theses, conference proceedings, journal papers and internet materials. The information involves interpretation of existing facts to establish logical reasoning and analysis leading to conclusion and recommendations. The rules are established using differentiation from first principles; and are then employed to find derivatives of some functions so as to confirm their applicability.

III. RESULTS AND DISCUSSIONS

In this section, the rules of differentiation are derived by the fundamental method-Differentiation from first principles. Above all, the constant, multiple constant, sum, difference, product, quotient, reciprocal, power and chain rules are considered. The rules and the fundamental method are then employed to calculate the derivatives of some selected functions. In fact, this implies the derivative of each selected function is calculated using these methods; and this is done just for substantiation of applicability.

A The Rules of Differentiation

The rules are introduced on the assumptions that the functions $f(x)$, $g(x)$ and $h(x)$ are defined on $[\alpha,\beta]$, continuous in an open interval (α,β) and differentiable in a closed interval $[\alpha,\beta]$. Thus, we have

The Constant Rule

Theorem 3: *Let $f(x)=k$ be a differentiable function and let k be a constant. Then the derivative of $f(x)$ is 0.*

This means that

$$\frac{d}{dx}\left(f(x)\right)=\frac{d}{dx}(k)=0$$

In Lagrange notation, this can be expressed as

$$f'(x)=0$$

Proof: Since this is a constant function then, for any x and Δx, we have

$$f(x+\Delta x)-f(x)=c-c=0$$

Thus, the derivative of $f(x)=k$ is given by

$$\frac{d}{dx}\left(f(x)\right)=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{c-c}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}(0)=0$$

Thus, we have

$$f'(x)=0 \text{ or } \frac{df}{dx}=0. \qquad (1)$$

The result (1) means the derivative of a constant function is zero.

The Constant Multiple Rule

Theorem 4 : *Let $f(x)$ be a differentiable function and let k be a constant. Then the derivative of $kf(x)$ is $kf'(x)$.*

That is,

$$\frac{d}{dx}\left[kf(x)\right]=kf'(x)$$

In Leibniz notation, this is expressed as

$$\frac{d}{dx}(kf)=k\frac{df}{dx}$$

Proof: The derivative of $kf(x)$ is given by

$$\frac{d}{dx}\left[kf(x)\right]=\lim_{\Delta x\to 0}\left[\frac{kf(x+\Delta x)-kf(x)}{\Delta x}\right]$$

$$=k\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

By the definition of the derivative, we have

$$\frac{d}{dx}\left[kf(x)\right]=k\frac{d}{dx}\left[f(x)\right]$$

In Lagrange's notation, this can be expressed as

$$\frac{d}{dx}\left(k\left(f(x)\right)\right)=kf'(x) \qquad (2)$$

The result (2) means the derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.

The Sum Rule

Theorem 5: *Let $g(x)$ and $h(x)$ be differentiable functions where $f(x)=g(x)+h(x)$. Then the derivative of $f(x)$ is $g'(x)+h'(x)$.*

That is,

$$\frac{d\left[f(x)\right]}{dx}=\frac{d}{dx}\left[g(x)+h(x)\right]=g'(x)+h'(x)$$

In Leibniz notation, this statement can be written as

$$\frac{df}{dx}=\frac{dg}{dx}+\frac{dh}{dx}$$

Proof: The derivative of $g(x)+h(x)$ is given by.

$$f'(x)=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)+h(x+\Delta x)-(g(x)+h(x))}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)+h(x+\Delta x)-g(x)-h(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)+h(x+\Delta x)-h(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{(g(x+\Delta x)-g(x))+(h(x+\Delta x)-h(x))}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]+\lim_{\Delta x\to 0}\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right]$$

By the definition of derivative, we have

$$f'(x)=g'(x)+h'(x)$$

In Leibniz notation, this is expressed as

$$\frac{df}{dx} = \frac{dg}{dx} + \frac{dh}{dx} \qquad (3)$$

The result (3) means the derivative of the sum of two functions is equal to the sum of the derivatives of the functions.

The Difference Rule

Theorem 6: *Suppose* $f(x)$, $g(x)$ *and* $h(x)$ *are any differentiable functions, which are related in the following manner:* $f(x) = g(x) - h(x)$. *.Then the derivative of* $f(x)$ *is* $g'(x) - h'(x)$

This means

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[g(x) - h(x)] = g'(x) - h'(x)$$

In Leibniz notation, this is expressed as

$$\frac{df}{dx} = \frac{dg}{dx} + \frac{dh}{dx}$$

Proof: The derivative of $g(x) - h(x)$ is given by.

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - h(x+\Delta x)) - (g(x) - h(x))}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x) - h(x+\Delta x) - g(x) + h(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - g(x)) - (h(x+\Delta x) - h(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] - \lim_{\Delta x \to 0}\left[\frac{h(x+\Delta x) - h(x)}{\Delta x}\right]$$

$$f'(x) = g'(x) - h'(x)$$

Thus, we have

$$\frac{df}{dx} = \frac{dg}{dx} - \frac{dh}{dx} \qquad (4)$$

The result (4) means the derivative of the difference of two functions is equal to the difference of the derivatives of the constituent functions.

The Product Rule

Theorem: 7 *Suppose* $f(x)$, $g(x)$ *and* $h(x)$ *are any differentiable functions, which are related in the following:* $f(x) = g(x)h(x)$ *.Then the derivative of* $f(x)$ *is* $g'(x)h(x) + g(x)h'(x)$

This means that

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[g(x)h(x)]$$

$$= h(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[h(x)]$$

$$= h(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[h(x)]$$

$$= g'(x)h(x) + g(x)h'(x)$$

In Leibniz notation, we have

$$\frac{df}{dx} = \frac{dg}{dx}h + g\frac{dh}{dx}$$

Proof: Then the derivative of $g(x)h(x)$ is given by.

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

By the definition of the derivative, we have

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - g(x))h(x+\Delta x) + (h(x+\Delta x) - h(x))g(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - g(x))h(x+\Delta x)}{\Delta x}\right] + \lim_{\Delta x \to 0}\left[\frac{g(x)(h(x+\Delta x) - h(x))}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] \times \lim_{\Delta x \to 0}(h(x)) + \lim_{\Delta x \to 0}(g(x)) \times \lim_{\Delta x \to 0}\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right]$$

By the definition of the derivative, we have

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$

Thus, we have

$$\frac{df}{dx} = \frac{dg}{dx}h + g\frac{dh}{dx} \qquad (5)$$

The result (5) means the derivative of the product of two functions is equal to the sum of the derivative of the first function multiplied by the second function and the derivative of the second function multiplied by the first function.

The Quotient Rule

Theorem 8: *Suppose* $f(x)$, $g(x)$ *and* $h(x)$ *are differentiable functions where* $f(x) = g(x)/h(x)$. *Then the derivative of* $f(x)$ *is*

$$[g(x)'h(x) - g(x)h'(x)]/[h(x)]^2$$

That is,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{g(x)}{h(x)}\right]$$

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

In Leibniz notation, this is expressed as

$$\frac{df}{dx} = \frac{h\frac{dg}{dx} - g\frac{dh}{dx}}{h^2}$$

Proof: Then the derivative of $f(x)$ at x is given by.

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\frac{g(x+\Delta x)}{h(x+\Delta x)} - \frac{g(x)}{h(x)}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x)h(x) - g(x)h(x+\Delta x)}{h(x+\Delta x)h(x)\Delta x}\right]$$

This becomes

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x)h(x) - g(x)h(x) + g(x)h(x) - g(x)h(x+\Delta x)}{h(x+\Delta x)h(x)\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - g(x))h(x) - g(x)(h(x+\Delta x) - h(x))}{h(x+\Delta x)h(x)\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - g(x))h(x)}{h(x+\Delta x)h(x)\Delta x}\right] - \lim_{x \to \Delta x}\left[\frac{g(x)(h(x+\Delta x) - h(x))}{h(x+\Delta x)h(x)\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - g(x))}{\Delta x}\right] \times \lim_{x \to \Delta x}\left(\frac{h(x)}{h(x+\Delta x)h(x)}\right) - \lim_{\Delta x \to 0}\left(\frac{g(x)}{h(x+\Delta x)h(x)}\right) \times \lim_{\Delta x \to 0}\left[\frac{h(x+\Delta x) - h(x)}{\Delta x}\right]$$

By the definition of the derivative, we have

$$f'(x) = \frac{g'(x)h(x)}{[h(x)]^2} - \frac{g(x)h'(x)}{[h(x)]^2}$$

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

In Leibniz notation, this is expressed as

$$\frac{df}{dx} = \frac{h\frac{dg}{dx} - g\frac{dh}{dx}}{h^2} \qquad (6)$$

The result (6) means the derivative of a quotient of functions is equal to the denominator function times the derivative of the enumerator function minus the numerator function times the derivative of the denominator function; all divided by the square of the denominator function [13].

The Reciprocal Rule

Theorem 9: Let $f(x)$ and $g(x)$ are differentiable functions where $f(x) = 1/g(x)$. Then the derivative of $f(x)$ is $[-g(x)']/[h(x)]^2$

This means that

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{1}{g(x)}\right] = \frac{-g'(x)}{[g(x)]^2}$$

In Leibniz notation, this is written as

$$\frac{df}{dx} = \frac{-\frac{dg}{dx}}{g^2}$$

Proof: Then the derivative of $\frac{1}{g(x)}$ is given by.

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{g(x) - g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{-(g(x+\Delta x) - g(x))}{g(x+\Delta x)g(x)\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(g(x+\Delta x) - g(x))}{\Delta x}\right] \times \lim_{\Delta x \to 0}\left(\frac{-1}{g(x+\Delta x)g(x)}\right)$$

By the definition of the derivative, we have

$$f'(x) = g'(x) \times \frac{1}{[g(x)]^2} = \frac{-g'(x)}{[g(x)]^2}$$

$$f'(x) = \frac{-g'(x)}{[g(x)]^2}$$

In Leibniz notation, we have

$$\frac{df}{dx} = \frac{-\frac{dg}{dx}}{g^2} \qquad (7)$$

The analytical result (7) means that the derivative of the reciprocal of a function is the negative quotient of the derivative and the square of the function.

The Power Rule

Theorem 10: *Suppose* $f(x)$ *and* $g(x)$ *are differentiable functions such that* $f(x) = 1/g(x)$. *Then the derivative of* $f(x)$ *is* $[-g(x)']/[h(x)]^2$

Proof: The derivative of $1/g(x)$ is given by.

$$f(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{x^n + \frac{nx^{n-1}\Delta x}{1!} + \frac{n(n-1)x^{n-2}(\Delta x)^2}{2!} + \frac{n(n-1)(n-2)x^{n-3}(\Delta x)^3}{3!} \ldots - x^n}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left(nx^{n-1} + \frac{n(n-1)x^{n-2}\Delta x}{2!} + \frac{n(n-1)(n-2)x^{n-3}(\Delta x)^2 \ldots}{3}\right)$$

$$f'(x) = nx^{n-1}$$

In Leibniz notation, this is expressed as

$$\frac{df}{dx} = nx^{n-1} \qquad (8)$$

The result (8) means that the derivative of a variable base raised to a constant power is the power multiplied by the variable base raised to the power minus one [13].

The Chain Rule

Theorem 11: *Let* $f(x)$ *and* $g(x)$ *be differentiable functions where* $y = f(g(x))$. *Then the derivative of the composition* $f(g(x))$ *is* $f'(g(x)) \times g'(x)$

This means

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))] = \frac{d}{dx}[f(g(x))] \times \frac{d}{dx}[g(x)]$$

$$y' = f'(g(x))g'(x)$$

It is deduced that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f' \times g' = f'g'$$

Proof: To calculate the derivative of $f(g(x))$, we must evaluate the following limit

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))]$$

$$= \lim_{\Delta x \to 0}\left[\frac{f(g(x+\Delta x) - f(g(x))}{\Delta x}\right]$$

But the expression $\frac{f(g(x+\Delta x) - f(g(x))}{\Delta x}$ can trickily be written as

$$\frac{f(g(x+\Delta x) - f(g(x))}{g(x+\Delta x) - g(x)} \times \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Then $y = f(g(x))$ may be decomposed into $u = g(x)$ and $y = f(u)$, which can be illustrated diagrammatically as

$$x \xrightarrow{g} u \xrightarrow{f} y \backslash$$

Thus, we have

$$\frac{d}{dx}[f(g(x))] = \lim_{\Delta x \to 0}\left[\frac{f(g(x+\Delta x) - f(g(x))}{g(x+\Delta x) - g(x)}\right] \lim_{\Delta x}\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(u+\Delta u) - f(u)}{\Delta u}\right] \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right]$$

where $u = g(x)$ and $u + \Delta u = g(x + \Delta x)$

In functional notation, we have

$$\frac{d}{dx}[f(g(x))] = f'(u)g'(x)$$

$$= f'(g(x)g'(x) = f' \times g' = f'g'$$

In Leibniz notation, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad (9)$$

The result (9) means the derivative of a composite function is the derivative of the outer function calculated at the inner function multiplied by the derivative of the inner function [13].

B Numerical Experiments

Example1: Find the derivative of $f(x) = ax + bx$

SOLUTION

(a) Using the fundamental method, we have:

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{a(x+\Delta x) + b(x+\Delta x) - ax - bx}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{ax + a\Delta x + bx + b\Delta x - ax - bx}{\Delta x}\right]$$

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{a\Delta x + b\Delta x}{\Delta x}\right] = a + b$$

(b) Using the rule of sums, we have

$$\frac{d(f(x))}{dx} = \frac{d}{dx}(ax) + \frac{d}{dx}(bx) = a + b$$

$$\frac{dy}{dx} = a + b \qquad (y = f(x))$$

By using Lagrange’s notation, the derivative is expressed as

$$f'(x) = a + b$$

Example 2: Differentiate the function

$f(x) = 3x - \sin x$

SOLUTION

(a) Using the method of differentiation from first principles, we have

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{3(x+\Delta x) - \sin(x+\Delta x) - (3x - \sin x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{3x + 3\Delta x - \sin(x+\Delta x) - 3x + \sin x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{3\Delta x - (\sin(x+\Delta x) - \sin x)}{\Delta x}\right]$$

$$= 3 - \lim_{\Delta x \to 0}\left[\frac{\sin(x+\Delta x) - \sin x}{\Delta x}\right]$$

But $\sin(x+\Delta x) = \sin x \cos\Delta x + \cos x \sin\Delta x$

and so,

$$\lim_{\Delta x \to 0}\left[\frac{\sin(x+\Delta x) - \sin x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sin x(\cos\Delta x - 1) + \cos x \sin\Delta x}{\Delta x}\right]$$

$$= (\sin x)\lim_{\Delta x \to 0}\left[\frac{(\cos\Delta x - 1)}{\Delta x}\right]$$

$$+ (\cos x)\lim_{\Delta x \to 0}\left[\frac{\sin\Delta x}{\Delta x}\right]$$

Using the standard limits

$$\lim_{\Delta x \to 0}\left[\frac{(\cos\Delta x - 1)}{\Delta x}\right] = 0 \text{ and } \lim_{\Delta x \to 0}\left[\frac{\sin\Delta x}{\Delta x}\right] = 1$$

we have

$$\lim_{\Delta x \to 0}\left[\frac{\sin(x+\Delta x) - \sin x}{\Delta x}\right] = \cos x$$

Thus, $f'(x) = 3 - \cos x$

Example 3: Find the derivative of the function $f(x) = \sqrt{x}\cos x$

SOLUTION

(a) Using differentiation from first principles, the derivative of the function $f(x)$ is given by

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sqrt{(x+\Delta x)}\cos(x+\Delta x) - \sqrt{x}\cos x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sqrt{(x+\Delta x)}\cos(x+\Delta x) - \sqrt{x}\cos(x+\Delta x) + \sqrt{x}\cos(x+\Delta x) - \sqrt{x}\cos x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(\sqrt{x+\Delta x} - \sqrt{x})\cos(x+\Delta x) + (\cos(x+\Delta x) - \cos(x))\sqrt{x}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(\sqrt{x+\Delta x} - \sqrt{x})\cos(x+\Delta x)}{\Delta x}\right] + \lim_{\Delta x \to 0}\left[\frac{\sqrt{x}(\cos(x+\Delta x) - \cos(x))}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\right] \times \lim_{\Delta x \to 0}\left(\cos(x)\right) + \lim_{\Delta x \to 0}\left(\sqrt{x}\right) \times \lim_{\Delta x \to 0}\left[\frac{\cos(x+\Delta x) - \cos(x)}{\Delta x}\right]$$

But $\lim_{\Delta x \to 0}\left[\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{\left(\sqrt{x+\Delta x} - \sqrt{x}\right)\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}\right] = \lim_{\Delta x \to 0}\left(\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}\right)$

$$= \frac{1}{2\sqrt{x}}$$

and

$$\lim_{\Delta x\to 0}\left[\frac{\cos(x+\Delta x)-\cos(x)}{\Delta x}\right]=\lim_{\Delta x\to 0}\left[\frac{-2\sin(x+\frac{\Delta x}{2})\sin\frac{\Delta x}{2}}{\Delta x}\right]$$

$$=-\lim_{\Delta x\to 0}\left(\sin(x+\frac{\Delta x}{2})\right)\times\lim_{\Delta x\to 0}\left(\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right)=-\sin x$$

Hence $f'(x)=\frac{1}{2\sqrt{x}}\cos x-\sqrt{x}\sin x$

(b) Using the product rule, we have

$$\frac{d}{dx}\left(f(x)\right)=\frac{d}{dx}\left(\sqrt{x}\right)\cos x+\sqrt{x}\frac{d}{dx}\left(\cos x\right)$$

$$=\frac{1}{2\sqrt{x}}\cos x+\sqrt{x}\left(-\sin x\right)=\frac{1}{2\sqrt{x}}\cos x-\sqrt{x}\sin x$$

Thus $f'(x)=\frac{1}{2\sqrt{x}}\cos x-\sqrt{x}\sin x$

Example 4: Find the derivative of the function $f(x)=\frac{\sin x}{2x^2}$

SOLUTION

(a) Using differentiation from first principles we have

$$f'(x)=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{\frac{\sin(x+\Delta x)}{2(x+\Delta x)^2}-\frac{\sin x}{2x^2}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{2x^2\sin(x+\Delta x)-2(x+\Delta x)^2\sin x}{4x^2(x+\Delta x)^2\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{2x^2\sin(x+\Delta x)-2x^2\sin(x)+2x^2\sin x-2(x+\Delta x)^2\sin x}{4x^2(x+\Delta x)^2\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{2x^2(\sin(x+\Delta x)-\sin(x))-(2(x+\Delta x)^2-2x^2)\sin x}{4x^2(x+\Delta x)^2\Delta x}\right]$$

But $\lim_{\Delta x\to 0}\left[\frac{(\sin(x+\Delta x)-\sin(x))}{\Delta x}\right]=\cos x$ and

$$\lim_{\Delta x\to 0}\left[\frac{2(x+\Delta x)^2-2x^2}{\Delta x}\right]=2\lim_{\Delta x\to}\left[\frac{(x+\Delta x)^2-x^2}{\Delta x}\right]=4x$$

So, $\lim_{\Delta x\to 0}\left[\frac{2x^2(\sin(x+\Delta x)-\sin(x))-(2(x+\Delta x)^2-2x^2)\sin x}{4x^2(x+\Delta x)^2\Delta x}\right]=\frac{2x^2\cos x-4x\sin x}{4x^4}$

Thus, $f'(x)=\frac{2x^2\cos x-4x\sin x}{4x^4}$

(b) Using the rule of a quotient we have

$$\frac{d}{dx}\left(f(x)\right)=\frac{d}{dx}\left(\frac{\sin x}{2x^2}\right)$$

$$=\frac{2x\frac{d}{dx}\left(\sin x\right)-\sin x\frac{d}{dx}\left(2x^2\right)}{\left(2x^2\right)^2}$$

$$=\frac{2x\cos x-4x\sin x}{4x^4}$$

Thus, $f'(x)=\dfrac{2x^2\cos x-4x\sin x}{4x^4}$

Example 5: Find the derivative of the function

$$f(x)=\frac{1}{\log_e x}$$

SOLUTION

(a) Using differentiation from first principles, we have

$$f'(x)=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{\dfrac{1}{\log_e(x+\Delta x)}-\dfrac{1}{\log_e x}}{\Delta x}\right]$$

$$=-\lim_{\Delta x\to 0}\left[\frac{\log_e(x+\Delta x)-\log_e x}{\Delta x}\right]$$

$$\times\lim_{\Delta x\to 0}\left[\frac{1}{\log_e(x+\Delta x)\log_e x}\right]$$

$$=-\frac{1}{x}\times\frac{1}{(\log_e x)^2}=-\frac{1}{x(\log_e x)^2}$$

Thus $f'(x)=-\dfrac{1}{x(\log_e x)^2}$

(b) Using the reciprocal rule we have

$$\frac{d}{dx}\big(f(x)\big)=\frac{d}{dx}\left(\frac{1}{\log_e x}\right)=\frac{-\dfrac{d}{dx}(\log_e x)}{(\log_e x)^2}$$

$$=\frac{-\dfrac{1}{x}}{(\log_e x)^2}=-\frac{1}{x(\log_e x)^2}$$

So, $f'(x)=-\dfrac{1}{x(\log_e x)^2}$

Example 6: Differentiate the function $f(x)=\sqrt{e^x}$

SOLUTION

(a) Using the fundamental method, we have

$$f'(x)=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{\sqrt{e^{x+\Delta x}}-\sqrt{e^x}}{e^{x+\Delta x}-e^x}\right]\lim_{\Delta x\to 0}\left[\frac{e^{x+\Delta x}-e^x}{\Delta x}\right]$$

If $u=e^x$ then $u+\Delta u=e^{x+\Delta x}$. So, we have

$$\frac{d}{dx}\big(f(x)\big)=\lim_{\Delta x\to 0}\left[\frac{\sqrt{u+\Delta x}-\sqrt{u}}{\Delta u}\right]$$

$$\times\lim_{\Delta x\to 0}\left[\frac{e^{x+\Delta x}-e^x}{\Delta x}\right]$$

By the definition of a derivative, this becomes

$$\frac{d}{dx}\big(f(x)\big)=\frac{1}{2\sqrt{u}}\times e^x=\frac{e^x}{2\sqrt{e^x}}=\frac{\sqrt{e^x}}{2}$$

Thus $f'(x)=\dfrac{\sqrt{e^x}}{2}$

(a) Using the chain rule we have

Let $u=e^x$. Thus, $y=\sqrt{u}$. Then $\dfrac{du}{dx}=e^x$ and

$$\frac{dy}{du}=\frac{1}{2\sqrt{u}}=\frac{1}{2\sqrt{e^x}}$$

Thus, we have

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2\sqrt{e^x}}\times e^x=\frac{e^x}{2\sqrt{e^x}}=\frac{\sqrt{e^x}}{2}$$

So, $f'(x)=\dfrac{\sqrt{e^x}}{2}$

In the aforesaid section, it was shown how the rules of differentiation are obtained using differentiation from first principles. It was also shown that both the rules and the fundamental method can be used to find the derivatives of functions. Clearly, the application of the rules takes less amount of computation; and therefore, reduces chances of making computational inaccuracies.

CONCLUSION

The purpose of this work was to review some basic rules of differentiation. Only nine rules were reviewed with differentiation from first principles and applied to calculate the derivatives of some selected functions. The study primarily examined the constant, multiple constant, sum, difference, product, quotient, reciprocal and chain rules. This was successfully executed using differentiation from first principles, where some basic features instituting the differential calculus were deliberately adopted. Moreover, the study incorporated the derivative notations introduced by Gottfried Leibniz (1646-1716) and Joseph-Luis Lagrange (1736-1813). The analytical results instituted that all the rules of differentiation originate from the fundamental principle introduced by these famous mathematicians. Examples on finding derivatives of some selected functions were integrated in the study, where both methods were used for each case. The results indicated that the rules are considerably easier to manipulate than the basic method. That is, they take less amount of calculation. Thus, the application of the rules reduces possibilities of making mistakes when computations are performed. This study recommends that calculus learners should use both methods when finding the derivative of a function to fully understand the concept differentiation. It should be clear to them that the rules originated from the fundamental principle; and with them computations of derivatives are easier.

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