Laminar Convection in a Uniformly Heated Vertical Elliptical Cylinder

DR. SARVESH NIGAM
Assistant Professor, Department of Mathematics, Pt. J. L. N. P. G. College Banda (U.P.)

Abstract- Here we have obtained solutions for fully developed laminar flow through a vertical cylinderwhose cross section is confocal vertical elliptical cylinder under a pressure gradient. The solutions are established in terms of Mathieu functions.

I. INTRODUCTION

Heat transfer problem of combined free and forced convection due to a fully developed laminar flow with constant wall temperature has been investigated for many years. On the other hand the situation with varying wall temperature has been studied only recently.

The problem of fully developed laminar convection flows of incompressible viscous fluid under a pressure gradient in a vertical circular cylinder with varying wall temperature was solved by Tao (1) and Morton (2). Dalip Singh (3) discussed the flows of incompressible viscous fluid under a pressure gradient in a vertical elliptical cylinder.

Here we have obtained solutions for fully developed laminar flow through a vertical cylinder whose cross-section is confocal vertical elliptical cylinder under a pressure gradient. The solutions are established in terms of Mathieu functions.

1. FORMULATION OF PROBLEM :

The flow is assumed to be developed steady and incompressible and to have constant physical properties except density. Jaking Z-axis along the axis of the cylinder the equation of continuity momentum for fully developed flow of incompressible viscous fluid in confocal vertical elliptical cylinder of linearly confocal vertical elliptical cylinder of linearly varying wall temperature with heat sources are (Tao, 1).

$$\frac{\partial u}{\partial z} = 0 \tag{1.1}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\rho g \beta t}{\mu} = \frac{1}{\mu}\left[\frac{\partial p}{\partial z} + (\rho_w - p_\circ)g\right] \tag{1.2}$$

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\rho C_p C_1 u}{K} = -\frac{Q}{K} \tag{1.3}$$

Where p is the pressure, ρ the density, μ the viscosity, g the acceleration due to gravity, β the expansivity, C_p the specific heat at constant pressure, K the thermal conductivity, Q the heat source intensity, and $C_1$the wall temperature gradient, μ is the axial velocity and $t$the difference of local and wall temperature.

Following Tao the dimensionless form of (1.1), (1.2) and (1.3) are

$$\frac{\partial u}{\partial z} = 0 \tag{1.4}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + R_a \phi = E \tag{1.5}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - U = -F \tag{1.6}$$

Where $U = \frac{u}{u_m}$

$$R_a = \frac{\rho^2 g C_p C_1 \beta \sigma^4}{K\mu},$$

$$\phi = \frac{Kt}{\rho C_p C_1 u_m \sigma^2},$$

$$F = \frac{Q}{\rho C_p C_1 u_m},$$

$$E = \frac{\sigma^2}{\mu u_m}\left[\frac{\partial p}{\partial z} + (\rho_m - \rho_\circ)g\right] \tag{1.7}$$

$R_a$being the Railegh number U_m the average velocity.

Again combining (1.5) and (1.6) with the help of complex function.

$$\theta = U + i\epsilon^2\phi$$

$$G = F + \frac{iE}{\epsilon^2} \qquad (1.8)$$

and $\epsilon^4 = R_a$

we get

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - i\epsilon^2\theta = -i\epsilon^2 G \qquad (1.9)$$

Let $h^2 = i^3\epsilon^2$

So equation (1.9) transforms to

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + h^2\theta = h^2 G \qquad (1.10)$$

Now let us introduce elliptic – Coordinates given by

$$x = C\cosh 2\xi - \cos\eta$$

$$y = C\ \sinh\xi\sin\eta \qquad (1.11)$$

In elliptic coordinates equation (1.10) transforms to

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} + 2K^2(\cosh 2\xi - \cos 2\eta)\theta = 2K^2 G(\cosh 2\xi - \cos 2\eta)$$

where $4K^2 = C^2h^2$ (1.12)

BOUNDARY CONDITIONS :

Let the boundary of the elliptic confocal cylinder is $\xi = \xi_\circ$ and $\xi = \xi_1$

So $\theta = 0$

where $\xi = \xi_\circ$

and $\xi = \xi_1$

SOLUTION :

Let solve equation (1.12) let us choose $q_{2n,m}$ to be the roots of the equation

$$C_{\rho 2n}(\xi_1, q)F_\rho y_{2n}(\xi_\circ, q) - F_\rho y_{2n}(\xi_1, q)C_{\rho 2n}(\xi_\circ, q) = 0.$$

Now multiply equation (1.12) by $\beta_{2n}(\xi, q_{2n,m})C_{2n}(\eta, q_{2n,m})$ and integrate ξ with in the limits $\xi_\circ$ to ξ_1 and η with in the limits $\circ$ to $2n$ where

$$\beta_{2n}(\xi, q_{2n,m}) = [\{F_\rho y_{2n}(\xi_\circ, q_{2n,m}) - F_\rho y_{2n}(\xi, q_{2n,m})\}C_{\rho 2n}(\xi, q_{2n,m}) - \{C_{\rho 2n}(\xi_\circ, q_{2n,m}) - C\rho 2n\xi 1,q2n,mF\rho y2n\xi,q2n,m \qquad (1.13)$$

We get

$$\int_{\xi_\circ}^{\xi_1}\int_\circ^{2\Pi}\left(\frac{\partial^2\theta}{\partial\xi^2} + \partial 2\theta\partial\eta 2\beta 2n\xi,q2n,mC\rho 2n\eta,q2n,md\xi d\eta\right.$$

$$+2K^2\int_{\xi_\circ}^{\xi_1}\int_\circ^{2\Pi}(\cosh 2\xi - \cos 2\eta\theta\beta 2n\xi,q2n,mC\rho 2n\eta,q2n,md\xi d\eta$$

$$= 2K^2 G\int_{\xi_\circ}^{\xi_1}\int_\circ^{2\Pi}(\cosh 2\xi - \cos 2\eta\beta 2n\xi,q2n,mC\rho 2n\eta,q2n,md\xi d\eta$$

or

$$-2q_{2n,m}\bar\theta + 2K^2\bar\theta = 2K^2 G\int_{\xi_\circ}^{\xi_1}\int_\circ^{2\Pi}(\cosh 2\xi - \cos 2\eta\beta 2n\xi,q2n,mC\rho 2n\eta,q2n,md\xi d\eta$$

or $$\bar\theta = \frac{K^2 G}{K^2 - q_{2n,m}}\int_{\xi_\circ}^{\xi_1}\int_\circ^{2\Pi}(\cosh 2\xi - \cos 2\eta\beta 2n\xi,q2n,mC\rho 2n\eta,q2n,md\xi d\eta.$$

By inversion theorem of Gupta (4), we get

$$\theta = \sum_{n=0}^{\infty}\frac{\left[\sum_{n=0}^{\infty}\beta_{2n}(\xi, q_{2n,m})C_{\rho 2n}(\eta, q_{2n,m})K^2 G\int_{\xi_\circ}^{\xi_1}\int_\circ^{2\Pi}(\cosh 2\xi - \cos 2\eta)\beta_{2n}(\xi, q_{2n,m})C_{\rho 2n}(\eta, q_{2n,m})d\xi d\eta\right]}{\Pi(K^2 - q_{2n,m})\int_{\xi_\circ}^{\xi_1}\beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta_{2n,m}]d\xi}$$

$$= G\sum_{n=0}^{\infty}\frac{\sum_{m=0}^{\infty}K^2\beta_{2n}(\xi, q_{2n,m})C_{\ell 2n}(\eta, q_{2n,m})\int_{\xi_\circ}^{\xi_1}\beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n}\cosh 2\xi - A_2^{2n}]d\xi}{\Pi(K^2 - q_{2n,m})\int_{\xi_\circ}^{\xi_1}A_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta_{2n,m}]d\xi}$$

TRANSITION TO CIRCULAR CYLINDERS :

We find as in Gupta (4)

$$C_{\rho 2n}(\xi, q_{2n,m}) \to P'_{2n}J_{2n}\left(\sqrt{\frac{\alpha}{\kappa}}\gamma\right)$$

$$F_\rho y_{2n}(\xi, q_{2n,m}) \to P'_{2n}Y_{2n}\left(\sqrt{\frac{\alpha}{\kappa}}\gamma\right)$$

$A_\circ^2 \to 2^{1/2}$, $A_\circ^2 \to \circ$, $\Theta_{2n} \to \circ$

$C_{\rho\circ}(\eta, q_{\circ,m}) \to 2^{-1/2}\alpha = \kappa p^2$.

By Gupta (5) equation (1.15) transforms to

$$\theta = -G\sum_p \frac{K^2}{p^2 - K^2}\frac{J_\circ^2(pb)\beta_\circ(p\gamma)}{J_\circ^2(pb) - J_\circ^2(pa)} \times \frac{J_\circ(pb) - J_\circ(pa)}{J_\circ(pb)} \qquad (1.16)$$

where $\beta_\circ(p\gamma) = J_\circ(p\gamma)\gamma_\circ(pa) - \gamma_\circ(p\gamma)J_\circ(pa)$

Where $p's$ are the roots of the equation.

$J_\circ(pb)\gamma_\circ(pa) - \gamma_\circ(pb)J_\circ(pa) = 0$ (1.17)

$$\sum_{n=0}^{\infty} \frac{\sum_{m=1}^{\infty} iC^2 \in^2 \beta_{2n}(\xi, q_{2n,m}) C_{\rho 2n}(\eta, q_{2n,m}) \times \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi}{\Pi(C^2\epsilon^2 i + \Theta q_{2n,m}) \int_{\xi_\circ}^{\xi_1} A_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

$$= \sum_{n=1}^{\infty} \frac{\sum_{m=1}^{\infty} \beta_2(\xi, q_{2n,m}) C_{\rho 2n}(\eta, q_{2n,m}) \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m}) \times [2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi\, C^2(\in^4 PC^2 - 4q_{2n,m}E) + i\in^2 C^2(4Fq_{2n,m} + \in^4 C^2 E)}{\Pi(C^4\in^4 + 16q_{2n,m}^2) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

.

On separating real and imaginary parts, we get

$$\cup = -\sum_{n-1}^{\infty} \frac{\sum_{m-1}^{\infty} C^2(R_a FC^2 - 4q_{2n,m}E)\beta_{2n}(\xi, q_{2n,m}) C_{\ell 2n}(\eta, q_{2n,m}) \times \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi}{\Pi(16q_{2n,m}^2 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

(1.18)

$$\phi = -\sum_{n=0}^{\infty} \frac{C^2 \sum_{m=1}^{\infty} (C^2 E + 4q_{2n,m}F)\beta_{2n,m}(\xi, q_{2n,m}) C_{\ell 2n}(\eta, q_{2n,m}) \times \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi}{\Pi(16q_{2n,m}^2 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

(1.19)

Now $U = \frac{u}{u_m}$

So

$$4 = -\sum_{n=0}^{\infty} \frac{\sum_{m=1}^{\infty} u_m C^2(FC^2 - 4q_{2n,m}F)\beta_{2n}(\xi, q_{2n,m}) C_{\ell 2n}(\eta, q_{2n,m}) \times \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi}{\Pi(16q_{2n,m}^2 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

(1.20)

and $\phi = \frac{Kt}{\rho C_p C_1 u_m \sigma^2}$

$$t = \frac{\rho C_p C_1 u_m \sigma^2}{K}\phi$$

$$t = -\sum_{n=0}^{\infty} \frac{\rho C_p C_1 u_m \sigma^2 C^2 \sum_{m=1}^{\infty} (C^2 E + 4q_{2n,m})\beta_{2n}(\xi, q_{2n,m}) C_{\ell 2n}(\eta, q_{2n,m}) \times \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi}{\Pi K(16q_{2n,m}^2 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

(1.21)

Now to evaluate the flow rate we have

$$\int_A UdA = u_m \int_A dA$$

$$ee \frac{\sum_{m=1}^{\infty} C^2(FC^2 R_a - 4q_{2n,m}) \int_{\xi_\circ}^{\xi_1} \int_\circ^{2\Pi} \beta_{2n}(\xi, q_{2n,m}) C_{\rho 2n}(\eta, q_{2n,m}) \times \frac{C^2}{2}[\cosh 2\xi - \cos 2\eta]d\xi d\eta \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi}{\Pi(16q_{2n,m}^2 + \epsilon^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi} = \frac{\Pi C^2}{2}(\sinh 2\xi_1 - \sinh 2\xi_\circ)$$

Now $\cup + \boldsymbol{i} \in^2 \emptyset = -\left(\boldsymbol{F} + \frac{\boldsymbol{i}\in}{\in^2}\right)$

$$\sum_{n=0}^{\infty} \frac{\sum_{m=1}^{\infty} C^2(4q_{2n,m}E - \beta C^2 R_a)\left[\int_{\xi_\circ}^{\xi_1} P_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi\right]^2}{\Pi^2(16q_{2n,m}^1 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi} = (\sinh 2\xi_1 - \sinh 2\xi_\circ)$$

or

$$E \sum_{n=0}^{\infty} \frac{\sum_{m=1}^{\infty} 4C^2 q_{2n,m}\left[\int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi\right]^2}{\Pi^2(16q_{2n,m}^1 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

$$= \sum_{n=0}^{\infty} \frac{\frac{FC^4 R_a}{\Pi^2} \sum_{m=1}^{\infty} \left[\int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi\right]^2}{(16q_{2n,m}^2 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi} + (\sinh 2\xi_1 - \sinh 2\xi_\circ)$$

or $E = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{FC^2 R_a}{4q_{n,m}} + (\sinh 2\xi_1 - \sinh 2\xi_\circ)$

$$\times \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(16q_{2n,m}^2 + C^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}{4\Pi^2 C^2 q_{2n,m}\left[\int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})\{2A_\circ^{2n} \cosh 2\xi - A_2^{2n}\}d\xi\right]^2}$$

(1.22)

Now mixed temperature is given by

$$T_M = \frac{\int_\circ tudA}{\int_\circ udA}.$$

On substituting the values of t, u, dA and integrating and on making use of orthogonal property of Mathieu functions, we get

$$T_M = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\rho C_p \sigma^2 C^2(C^2\epsilon + 4q_{2n,m})(FC^2 R_a - 4q_{2n,m}) \times \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi \times \int_{\xi_\circ}^{\xi_1} \beta_{2n}(\xi, q_{2n,m})[2A_\circ^{2n} \cosh 2\xi - A_2^{2n}]d\xi}{(\sinh 2\xi_1 - \sinh 2\xi_\circ)\Pi^2 K(16q_{2n,m}^2 + \in^4 R_a) \int_{\xi_\circ}^{\xi_1} \beta_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta q_{2n,m}]d\xi}$$

(1.23)

Now the Nusselts number in this case is

$$N_u = \frac{\pi c^2/2(\sinh 2\xi_1 - \sinh 2\xi_\circ)}{4C(\cosh \xi_1 - \cosh \xi_\circ) \int_\circ^{\Pi/2} \sqrt{1 - e^2 \cos^2 \Theta d\Theta}} \cdot \frac{F-1}{T_M}$$

Where e is the eccentricity of the elliptical tube.

So $$N_u = \frac{\pi c\,(\sinh\ 2\xi_1 - \sinh\ 2\xi_\circ)}{8(\cosh\ \xi_1 - \cosh\ \xi_\circ)\int_\circ^{\Pi/2}\sqrt{1-e^2\cos^2\Theta d\Theta}} \times \frac{F-1}{T_M} \qquad (1.24)$$

Where T_M is given by Equation 1.23.

II. REMARKS

Here the solutions are in the form of double aeries involving Mathieu functions. The rapidity of convergence is an important point. It can be easily seen that first three terms of the series are sufficient to give the shape of the results.

III. DISCUSSIONS

It is seen that the results hold good only for positive Raileigh numbers. For zero or negative Raileigh number the formula obtained do not hold good.

In this the Mathieu functions and transform applicable to Mathieu functions analogous to Hankel transform are used.

REFERENCES

[1] Tao, L. N. (1960): Applied Science Res. Sec. Ag. 357.

[2] Morton, B. R. (1960: Journal Fluid Mech., 5, 227.

[3] Dalip Singh (1965) : Bulletin of the Calcutts, Mathematical Society, pp. 109 to 116.

[4] Gupta, R. K. (1964) : Proceeding of the National Institute of Sciences of India, Vol. 30, A. No. 6.

[5] Nigam , S.: Indian Journal I.R.E. journal of science,2017(IV)

[6] Nigam ,S.: Indian Journal I.R.E. journal of science,2017(III)

[7] Nigam ,S.: Indian Journal I.R.E. journal of science,2017(II)

[8] Nigam ,S.: Indian Journal I.R.E. journal of science,2017(V)