Solving Nonlinear Ordinary Differential Equation of Electric Power Flow Model Using Lie Symmetry Method

RHODA M. MAMULI[1], DR VINCENT N. MARANI[2], PROF. MICHAEL O. ODUOR[3]
[1] *Department of Mathematics, Kibabii University, Kenya.*
[2, 3] *School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, Bondo, Kenya*

Abstract- A differential equation symmetry is a change that joins any solution to another solution of the structure. Lie groups and their infinitesimal generators can be naturally prolonged to act on the space of independent variables. In this paper, we present analysis of Lie symmetry to solve a nonlinear ODE of an electric power flow model which is of form $P(n,m,m^{!},m^{!!},m^{!!!})=0$. The search study uses Lie Symmetry analysis method to transform the equation by subjecting it to an extension generator and obtain determining equations, by reducing equations to lower order and find a general solution of the third or- der nonlinear heat conduction from the invariance related potential system under scaling. We exploited the use of prolongations (extended transformations), infinitesimal genera- tors, variation of symmetries, adjoint symmetries, invariant transformation problems and integrating factors.

Indexed Terms- prolongations (extended transformations), infinitesimal generators, variation of symmetries, adjoint symmetries, invariant transformation problems and integrating factors.

I. INTRODUCTION

Nonlinear differential equations have established a substantial and expanded concerns to its wide submissions. Nonlinear ordinary differential equations perform an important part in numerous divisions of practicing and pure arithmetic and their submissions in practical engineering process [56]. By employing classical and nonclassical Lie symmetry analysis and some technical calculations, new infinitesimal generators were obtained which gave rise to derive the invariant solutions for this class of equations [5]. We subject the equation to prolongation to construct determining equations, which enables us to find the symmetries and n-one parameter symmetries.

II. LIE SYMMETRY ANALYSIS OF NONLINEAR THIRD ORDER ORDINARY DIFFERENTIAL EQUATION.

We are analyzing nonlinear ODE of third order. In this study we will consider power electric flow using the equation. $P(n,m,m^{!},m^{!!},m^{!!!})=0$ The equation can be solved using analytical method or numerical method which depends on initial and boundary conditions to approximation, given as

$$(m)m^{!!!}+(m^{!2}+1)^{-1}=(m+m^{1}) \quad (1)$$

and transformed to

$$m^{(3)}=P\ (n,\ m,\ m,^{!}m^{!!}) \quad (2)$$

Analytically using Lie symmetry analysis approach the equation can alternatively be:-

$$m^{!!!}=mm!+m(m^{!})^{-1}+m^{!2}+1 \quad (3)$$

In this case third prolongation, which from the n^{th} extension form is given by:-

$$S^{[3]}S^{[2]}+(\beta^{!!!}-3^{!!!}\alpha^{!}-3^{!!}\alpha^{!!}-3^{!}\alpha^{!!!})\frac{\partial}{\partial m^{!!!}}$$

$S^{[3]} = \alpha \frac{\partial}{\partial n} + \beta \frac{\partial}{\partial m} + (\beta^{!} - \alpha^{!} m^{!}) \frac{\partial}{\partial m^{!}} + (\beta^{!!} - 2^{!!}\alpha^{!} - m^{!}\alpha^{!\theta})$
$\frac{\partial}{\partial m^{!!}} + (\beta^{!!!} - \alpha^{!} - 3^{!!}3^{!!!}\alpha^{!!} - m^{!}\alpha^{!!!})$
(4)

Where S is a third ODE and, When $S^{[3]}$ acts on the differential equation it yields

$S^{[3]}[m^{!!!} - m(m^{!}) - m(m^{!})^{-1} - m^{!2} - 1] = 0$ (5)

Which is equivalent to

$\propto [m^4 - m^{!2} - mm^{!!} - 1 + mm^{!!}(m^{!})^{-2} - 2m^{!!}y^{!}]$
$+\beta(-m^{!} - (m^{!})^{-1})) + (\beta^{!} - \alpha^{!}m^{!})[-m + m(m^{!})^{-2}m^{!!} - 2m^{!}m^{!!}]$
$(\beta^{!!}2m^{!!}\alpha^{!} - m^{!}\alpha^{!!})[0] + (\beta^{!!!} - 3m^{!!!}\alpha^{!} - 3m^{!!}\alpha^{!!} - ym^{!}\alpha^{!!!})[1] = 0$ (6)

But then by substituting (6) and simplifying the results we obtain a simple equation of:

$-\beta m^{!} - \beta(m^{!})^{-1} + \beta^{!}m + \beta^{!}m(m^{!})^{-2}m^{!!} - 2\beta^{!}m^{!}m^{!!} + \alpha mm^{!!} - \alpha^{!}m(m^{!})^{-1}m^{!!} - 2\alpha^{!}(m^{!})^{2}m^{!!} + \beta^{!!!} - 3m^{!!!}\alpha^{!} - 3m^{!!}\alpha^{!!} - m^{!}\alpha^{!!!} = 0$

(7)

The primes in equation (7) refers to total derivatives and so the first, the second, the third total derivatives of α and β can be stated in terms of fractional derivatives i.e:-

$$\alpha = \frac{\partial\alpha}{\partial n} + \frac{\partial\alpha}{\partial m}\left\{from\ d(\alpha) = \left(\frac{\partial\alpha}{\partial n}\right)dn + \left(\frac{\partial\alpha}{\partial m}\right)dm\right\}$$

$$\alpha^{!!} = \frac{d}{dn}\left(\frac{\partial\alpha}{\partial n} + m^{!}\frac{\partial\alpha}{\partial m}\right) + \frac{d}{dm}\left(\frac{\partial\alpha}{\partial x} + m^{!}\frac{\partial\alpha}{\partial m}\right)m^{!}$$

$$= \frac{\partial^2\alpha}{\partial n^2} + m^{!}\frac{\partial^2\alpha}{\partial m\partial n} + m^{!!}\frac{\partial\alpha}{\partial m} + m^{!}\frac{\partial^2\alpha}{\partial n\partial m} + m^{!2}\frac{\partial^2\alpha}{\partial m^2} + 0$$

$$= \frac{\partial^2\alpha}{\partial n^2} + 2m^{!}\frac{\partial^2\alpha}{\partial m\partial n} + m^{!2}\frac{\partial^2\alpha}{\partial m^2} + m^{!!}\frac{\partial\alpha}{\partial m}$$

$$\alpha^{!!!} = \frac{\partial}{\partial n}\left(\frac{\partial^2\alpha}{\partial n^2} + 2m^{!}\frac{\partial^2\alpha}{\partial n\partial m} + m^{!!}\frac{\partial\alpha}{\partial} + m^{!2}\frac{\partial^2\alpha}{\partial m^2}\right)$$

$$+m^{!}\frac{\partial}{\partial m}\left(\frac{\partial^2\alpha}{\partial n^2 \propto} + 2m^{!}\frac{\partial^2\alpha}{\partial n\partial m} + m^{!!}\frac{\partial\alpha}{\partial m} + m^{!2}\frac{\partial^2\alpha}{\partial m^2}\right)$$

$$= \frac{\partial^3 \propto}{\partial n^3} + 2m^{!}\frac{\partial^3 \propto}{\partial n\partial n\partial m} + 2m^{!!}\frac{\partial^2 \propto}{\partial n\partial m} + m^{!!}\frac{\partial^2 \propto}{\partial n\partial m} + m^{!!!}\frac{\partial \propto}{\partial m}$$

$$+m^{!2}\frac{\partial^3\alpha}{\partial n\partial m^2} + 2m^{!}m^{!!}\frac{\partial^2\alpha}{\partial m^2} + m^{!}\frac{\partial^3\alpha}{\partial m\partial n^2} + 2m^{!2}\frac{\partial^3\alpha}{\partial m\partial n\partial y} + 0$$

$$+0 + m^{!}m^{!!}\frac{\partial^2\alpha}{\partial m^2} + 0 + m^{!3}\frac{\partial^3\alpha}{\partial m^3} + 0$$

$$= \frac{\partial^3\alpha}{\partial n^3} + 3m^{!}\frac{\partial^3\alpha}{\partial n^2\partial m} + 3m^{!!}\frac{\partial^2\alpha}{\partial n\partial m} + m^{!!!}\frac{\partial\alpha}{\partial m} + 3m^{!2}3m^{!2}\frac{\partial^3\alpha}{\partial n\partial m^2} +$$

$$3m^{!}m^{!!}\frac{\partial^2\alpha}{\partial m^2} + m^{!3}\frac{\partial^3\alpha}{\partial m^3} \qquad (8)$$

Which also applies to derivative β. And on substituting and expanding the derivatives equations this is what we get

$$-\beta m^{!} - \beta(m^{!})^{-1} - m\frac{\partial\beta}{\partial m} - mm^{!}\frac{\partial\beta}{\partial m} +$$

$$m(m^{-1})^{-2}m^{!!}\frac{\partial\beta}{\partial n} + mm^{1})^{-1}m^{!!}\frac{\partial\alpha}{\partial m}$$

$$-2m^{!2}m^{!!}\frac{\partial\alpha}{\partial n} - m^{!3}m^{!!}\frac{\partial\alpha}{\partial m} + m\frac{\partial^3\beta}{\partial n^3} + 3m^{!}\frac{\partial^3\beta}{\partial n^2\partial m} + 3m^{!!}\frac{\partial^2\beta}{\partial n\partial m} + mm^{!}\frac{\partial\beta}{\partial m}$$

$$mm^{-1}\frac{\partial\beta}{\partial m} + m^{!2}\frac{\partial\beta}{\partial m} + 3m^{!2}\frac{\partial^3\beta}{\partial n\partial m^2} + 3m^{!}m^{!!!}\frac{\partial^2\beta}{\partial m^2} + 3m^{!3}\frac{\partial^3\beta}{\partial m^3}$$

$$-3mm^{!}\frac{\partial\alpha}{\partial n} - 3mm^{t-1}\frac{\partial\alpha}{\partial n} - 3m^{!2}\frac{\partial\alpha}{\partial n} - 3\frac{\partial\alpha}{\partial n} - 3mm^{!2}\frac{\partial\alpha}{\partial m^2}$$

$$-3m\frac{\partial\alpha}{\partial m} - 3m^{!3}\frac{\partial\alpha}{\partial m} - 3m^{!}\frac{\partial\alpha}{\partial m} - 3m^{!}m^{!!}\frac{\partial^2\alpha}{\partial n\partial m} - mm^{!2}\frac{\partial\alpha}{\partial m} - m\frac{\partial\alpha}{\partial m}$$

$$-m^{!3}\frac{\partial\alpha}{\partial m} - m^{!}\frac{\partial\alpha}{\partial m} - 3m^{!3}\frac{\partial^3\alpha}{\partial n\partial y^2} - 3m^{!2}m^{!!}\frac{\partial^2\alpha}{\partial m^2} +$$

$$(m^{!})^4\frac{\partial^3\alpha}{\partial m^3} = 0 \qquad (9)$$

Equation (9) identifies and embraces randomly $n, m, m^{!}$ and $m^{!!}$ [16]. Since α and β are roles of n and m only, we then associate the constant powers of $m^{!}$, $m^{!!}$, and their combinations to zero. We get the following systems of unconstrained fractional differential equations known as determining equations ([16],[23],[38])

Sine $(m^{!})^{-1}m^{!!}: m\frac{\partial\beta}{\partial n} - m\frac{\partial\propto}{\partial n} = 0$, Then equation

$(m^{!})^2 m^{!!}: -2\frac{\partial\beta}{\partial m} - 2\frac{\partial\alpha}{\partial n} - 6\frac{\partial^2\alpha}{\partial m^2} = 0$ reduces to $\frac{\partial^2\propto}{\partial m^2} = 0$. Which when integrated we get:-

$$\frac{\partial^2\alpha}{\partial m^2} = 0 \Rrightarrow \frac{\partial^2\alpha}{\partial m^2} = 0,$$

$$\Rrightarrow \frac{\partial\alpha}{\partial y} = C_1 \Rrightarrow \propto = C_1 m + C_2$$

(10)

C_1 and C_2 are randomly functions of n. Substitute (10) and solve to find :-

$$(m^{!})^1 m^{!!}: -2\frac{\partial\beta}{\partial n} + 3\frac{\partial^2\beta}{\partial m^2} - 9\frac{\partial^2\alpha}{\partial n\partial m} = 0$$

$$3\frac{\partial^2\beta}{\partial m^2} - 9\frac{\partial^2\propto}{\partial n\partial m} = 0 \Rrightarrow \frac{\partial^2\beta}{\partial m^2} = 3C_1^{!}$$

$$(m^{!})^2 m^{!!}: -2\frac{\partial\beta}{\partial m} - 2\frac{\partial\alpha}{\partial n} - 6\frac{\partial^2\alpha}{\partial m^2} = 0$$

$$\frac{\partial\beta}{\partial m} + 3C_1^{!} y + C_3$$

$$\Rrightarrow \beta = \frac{3}{2}C_1^{!} m^2 + C_3 m + C_4$$

(11)

C_3 and C_4 are functions of n and by substituting equations (10) and (11)in the equation below we will have:-

$$\frac{\partial^2\alpha}{\partial m^2} = 0 \Rrightarrow \frac{\partial^2\alpha}{\partial m^2} = 0$$

$$\Rrightarrow \frac{\partial\alpha}{\partial m} = C_1 \Rrightarrow \propto C_1 y + C_2$$

$$3\frac{\partial^2\beta}{\partial m^2} - 9\frac{\partial^2\alpha}{\partial n\partial m} = 0 \Rrightarrow \frac{\partial^2\beta}{\partial m^2} = 3C_1^{!}$$

$$\frac{\partial\beta}{\partial m} + 3C_1^{!} m + C_4$$

$$(m^{!})^{-1} m^{!!}: m\frac{\partial\beta}{\partial m} - m\frac{\partial\alpha}{\partial n} = 0$$

$$\frac{\partial\beta}{\partial m} - \frac{\partial}{\propto}\partial n = 0 \Rrightarrow 3C_1^{!} m + C_3 - C_1^{!} m - C_2^{!} = 0$$

(12)

The powers of m to zero since we can figure out that C_1, C_2 and C_3 hang on n only hence giving :- $m^1: 2C_1^{!} - 0,$ $m^0: C_3 - C_2^{!} =$

(13)

By substituting equation (13) above into equation:-

$$(m^{!})^{-2} m^{!!}: -m\frac{\partial\beta}{\partial n} = 0$$

$$3\frac{\partial^2\beta}{\partial m^2} - 9\frac{\partial^2\alpha}{\partial n\partial m} = 0 \Rrightarrow \frac{\partial^2\beta}{\partial m^2} = 3C_1^{!}$$

$$\frac{\partial\beta}{\partial m} + 3C_1^{!} m + C_3$$

$$\Rrightarrow \beta = \frac{3}{2}C_1^{!} m^2 + C_3 m + C_4$$

Then the results will be:- $\frac{\partial\beta}{\partial n}0,$ $\Rrightarrow \frac{3}{2}C_1^{!!} y m^2 +$ $C_3^{!} m + C_4^{!} = 0$ (14)

And when we associate the aspects of powers of y to zero this are the results:-

$$m^2: C_1^{!!}$$

(15)

$$m^1: C_3^{!} =$$

(16)

$$m^{-1}: C_4^{!} = 0$$

(17)

Let us solve the differential equation form equation (17) to the following:-

$C_1^{!!} = \Rrightarrow C_1^{!!} = 0 \Rrightarrow C_1^{!} H_1 \Rrightarrow C_1 = H_{1x} + H_2$

(18)

And from equation (16) we have H as :- $C_3^{!} = 0 \Rrightarrow C_3 = H_3$ (19)

Considering equation (17) then it will be:- $C_4^{!} = \Rrightarrow C_4 = H_4$ (20)

Since $C_3 = H_3$, then in the equation $y^0: C_3 - C_2^{!} = 0$ will yield:-

$$C_3 - C_2^{!} = 0 \Rrightarrow C_2^{!} = C_3 \Rrightarrow C_2^{!} = H_{3x} + H_5$$

(21)

And so here $H_1, H_{2,} H_{3,} H_{4,} H_5$ are arbitrary constants and can be applied in equation and produce:-

$$\propto (x, y) = H_{1xy} + H_{2y} + H_{3x} + H_{5,}$$

$$\beta(n, ym) = \frac{3}{2}H_1 m^2 + H_{3m} + H_{4,}$$

(22)

We now apply (18) and (21) in (11) and an infinitesimal transformation is produced as:-

$$S = (H_1 nm + H_2 m + H_3 n + H_5)\frac{\partial}{\partial n} + (\frac{3}{2}H_1 m^2 + H_3 m + H_4)\frac{\partial}{\partial m} \quad (23)$$

Written simply as :−

$$S = H_1\left(nm\frac{\partial}{\partial n} + \frac{3}{2}m^2\frac{\partial}{\partial m}\right) + H_2\left(m\frac{\partial}{\partial n}\right) + H_3\left(n\frac{\partial}{\partial n} + m\frac{\partial}{\partial m}\right) + H_4\left(\frac{\partial}{\partial m}\right)$$

(24)

Which is a five parameter symmetry. For a third order there are maximum seven Lie point symmetry characterized in $m^{!!!} = 0.$ [6][31], by explicitly calculating the equations in symmetries in Lie brackets, we obtain the nonzero constant symmetries as

$[S_2, S_3] = S_1$

$[S_2, S_{34}] = S_2$

$[S_1, S_5] = S_3$

(25)

Now let us consider the three-dimensional sub-algebra

$$W_1 = \frac{\partial}{\partial x}$$
$$W_2 = \frac{\partial}{\partial y}$$
$$W_3 = \frac{\partial}{\partial x}$$

(26)

Now let us use the prolongations here

$$S^{[0]} = \alpha\frac{\partial}{\partial n} + \frac{\partial}{\partial m}$$
$$S^{[1]} = G + (\beta' - m'\alpha')\frac{\partial}{\partial m'}$$
$$S^{[2]} = G^{[1]} + (\beta'' - 2m''\alpha' - m'\alpha'')$$
$$S^{[3]} = G^{[2]} + (\beta'''3m'''\alpha' - 3m''\alpha'' - m'\alpha''')\frac{\partial}{\partial m'''}$$

(27)

The third order prolongation for operator $W_1 = \frac{\partial}{\partial n}$ is obtained as follows;-

$$W_1^{[0]} = 1.\frac{\partial}{\partial n} + 0.\frac{\partial}{\partial m} = \frac{\partial}{\partial n}$$
$$W^{[1]} = W_0^{[1]} + (0 - m'.0)\frac{\partial}{\partial m'} = \frac{\partial}{\partial n}$$
$$W^{[2]} = W_0^{[1]} + (0 - 2m''.0 - m)\frac{\partial}{\partial m''} = \frac{\partial}{\partial n}$$
$$W^{[3]} = W_0^{[2]} + \left(0 - 3m^{'''.0-3m^{'''.0-m'.0}}\right)\frac{\partial}{\partial m'''} = \frac{\partial}{\partial n}$$
$$W_0^{[3]} = 1.\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

(28)

Invariant $y = v$ is acquired order to integrate equation (3).

And for operator $W_3 = m\frac{\partial}{\partial n}$ the required third order prolongation will be as follows:-

$$W_1^{[0]} = m.\frac{\partial}{\partial n} + 0.\frac{\partial}{\partial m} = m\frac{\partial}{\partial n}$$
$$W_3^{[1]} = W_3^{[1]} + (m' - m'.0)\frac{\partial}{\partial m'} = m\frac{\partial}{\partial n} + m'\frac{\partial}{\partial m'}$$
$$W_3^{[2]} = W_3^{[0]} + (m'' - 2m''.0 - m'.0)\frac{\partial}{\partial m''}$$
$$= m\frac{\partial}{\partial n} + m'\frac{\partial}{\partial m'} + m''\frac{\partial}{\partial m''}$$
$$W_3^{[3]} = W_3^{[3]} + (m''' - 3m'''.0 - 3m''.0 - m'')\frac{\partial}{\partial m'''}$$
$$= m\frac{\partial}{\partial n} + m'\frac{\partial}{\partial m'} + m''\frac{\partial}{\partial m''}$$
$$W_3^{[3]} = m\frac{\partial}{\partial n} + m'\frac{\partial}{\partial m'}m''\frac{\partial}{\partial m''} + m'''\frac{\partial}{\partial my'''}$$

(29)

In order to integrate equation (3) then $\frac{dn}{m}\frac{dm'}{m'} \Rightarrow n = mInk_1m'$ where $k_1$is a constant and $\frac{dm'}{m'} = \frac{dm''}{m''} \Rightarrow Inm' = Ink_2m'' \Rightarrow k_2\frac{m'}{m''} \Rightarrow v_1 = \frac{m''}{m'}$ Where k_2 and v_1 are constants.

Let us now reduce equation (3) to a differential equation using Lie algebra to a lower differential equation. We will engage the invariant differentiation [37.]

$\frac{dy'}{m'} = \frac{dm''}{m''} \Rightarrow Inm' = Ink_2m'' \Rightarrow k_2\frac{m'}{m''} \Rightarrow v_1 = \frac{m''}{m'}$

That is $m = v, v_1 = \frac{m''}{m'}$

Where equation (3) decreases first order ODE as follows.[4][6]

$$\frac{dv}{dv} = vv'^{-1} + vv'^{-3} + 1 + v'^{-2} - v''\psi'^{-2} \qquad (30)$$

And hence the integrating factor is

$$\frac{dv}{du} + (u''u^{-1})v = uu'^{-1} + uu'^{-3}1 + u'^{-2} \qquad (31)$$

Let $P(u) = u''u^{-2}v$ and $Q(u) = uu'^{-1} + uu'^{-3} + 1u'^{-2}$ then our equation (31) has the formula:- $\frac{v}{du} + P(u)v = Q(u)$. Which is first order linear equation.

Hence equation(3)reduces to first order linear equation (31) which is integrable and direct.

If the integrating aspect is $I(u)$.

Then $I(u) = e^{f(u''u^{-2})}du = e^{-u'^{-1}}$

(32)

Therefore then

$$v = \frac{1}{e^{-u'^{-1}}}\int(e^{-u'^{-1}})(uu'^{-1} + uu'^{-3}1 + u'^{-2})du$$

Is the general solution in quadrature of the nonlinear power flow equation of equation (1)

CONCLUSION

In this paper, Constructed the characteristics equations and used invariant differentiation to reduce it into quadrature solved to the general solution.

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