Solving A Square Root Problem Using the Pierce Continued Fraction

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Abstract— In this paper, we have developed a new continued fraction using the pierce expansion which is called the pierce continued fraction. There are several applications of continued fractions and one important application is the approximation to the square root problem. In our work, we have introduced new convergent sequences to solve this problem using pierce continued fraction. The convergent rate of these two continued fractions is different from each other. Furthermore, an algorithm was developed using matlab programmed to find the pierce continued fraction. As a future work we hope to construct a new approach to solve the pelle's equation.

Indexed Terms—Approximation to square root problem, continued fraction, convergent, pierce expansion, pierce continued fraction.

I. INTRODUCTION

Continued fraction expansions of numbers are contained in the Euclidian algorithm [3] and are important in giving rational approximation of real numbers. They have been used since the beginning of science in many ancient civilizations including Greek, Indian, and Chinese [3]. The famous golden ration can be represented in an attractive manner using the corresponding continued fractions.

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ldots . 1 + \cfrac{1}{1}}}}}$$

There are several applications based on continued fractions in Linear Algebra [1], Group Theory [4], Convex Analysis [2], Design Theory [2], Approximation [2], solving linear Diophantine equations [2], and solving Pelle's equation [2]. One of the most important aspects of these continued fractions is their convergent which was introduced by the Dutch mathematician /astronomer Christiaan Huygens (1629-1695).

Definition 1.1 Continued Fraction [1]

Let *x* be a any real number,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ldots . a_{n-1} + \cfrac{1}{a_n}}}}}$$

where $a_1, a_2, \ldots a_n$ are natural numbers with $a_1, a_2, \ldots a_n$ positive is called a finite continued fraction.

Definition 1.2 (Convergent of a Continued Fraction) [2]

The continued fraction $[a_0, a_1, \ldots\ldots a_n]$ where k is a non-negative integer is called the nth convergent of the simple continued fraction $[a_0, a_1, \ldots\ldots a_n]$ and is denoted by C_n.

When solving the approximation square root problem, the convergent rate of the continued fraction is relatively slower. Therefore, a new continued fraction has been introduced using the Pierce expansion.

Definition 1.3 (Pierce Expansion)

Any real number $x, 0 < x \leq 1$, can be written as

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} - \ldots + \frac{(-1)^{n+1}}{a_1 a_2 a_3 \ldots\ldots a_n},$$

where a_1, a_2, a_3 are natural numbers

Only disadvantage of this representation is x should be $0 < x \leq 1$.

Using the above expansion, a new continued fraction, called the pierce continued fraction, can be obtained.

pierce continued fraction are defined for real numbers $x; 0 < x \leq 1$.

Definition 1.4 Pierce Continued Fraction

Any real number $x, 0 < x \leq 1$,

$$x=\cfrac{1-\cfrac{1-\cfrac{1-\cfrac{1-\cdot^{\cdot^{\cdot}}1-\frac{1}{a_3}}{a_4}}{a_3}}{a_2}}{a_1}=\{a_1,a_2,\ldots\ldots a_n\}$$ where $a_1, a_2, \ldots a_n$ are positive real numbers is called a finite pierce continued fraction. If the positive real numbers are integers then above finite pierce continued fraction is called a simple finite pierce continued fraction. The above simple finite pierce continued fraction is denoted by $\{a_1, a_2, \ldots\ldots a_n\}$. $a_1, a_2, \ldots a_n$ are called partial quotients of a simple finite pierce continued fraction.

When the number is greater than 1, (say) n, we take x as $x = n - [n]$.

Example 1

$$\frac{3}{8}=\cfrac{1-\cfrac{1-\frac{1}{2}}{1}}{2}=\{2,1,2\}$$

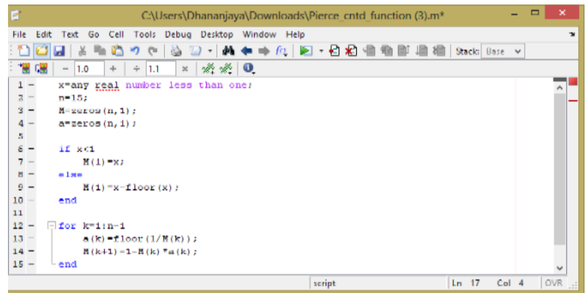
A new mapping has been proposed to construct the pierce continued fraction,

Define a mapping h from the interval $(0,1]$ to the set of positive real numbers, given by

$$h(x) = 1 - x\left(\left[\frac{1}{x}\right] - 1\right).$$

Set$a_m = \left\lfloor \frac{1}{h^{(k-1)}(x)} \right\rfloor$, where $h^{(m)}(x) = h\left(h^{(m-1)}(x)\right)$ and $h^{(0)}(x) = x$. (Here $\lceil . \rceil$and$\lfloor . \rfloor$ denote greatest integer less than or equal to '.', least integer greater than or equal to '.' respectively.) Using this we can compute the corresponding partial quotients.

Further, a MATLAB programmed has been developed to find the pierce continued fraction using above mapping as below:



For example, pierce continued fraction of $1/e$ is, $\{2,3,4,5,6,7\ldots\}$

Theorem 1.1

i. Every rational number x, $0 < x < 1$ less than 1 can be written as a simple pierce continued fraction.

ii. Every simple finite pierce continued fraction represent a rational number.

Proof

i. Clearly by the definition of pierce expansion.

ii. By using the Principle of Mathematical Induction

Consider $[a_0, a_1, \ldots\ldots a_n]$

When n=1,

$$[a_0, a_1] = \frac{1-\frac{1}{a_1}}{a_0} = \frac{\frac{a_1-1}{a_1}}{a_0} = \frac{a_1-1}{a_0 a_1} \in \mathbb{Q}$$

Assume that $[a_0, a_1, \ldots\ldots a_k]$ is a rational number $\frac{p}{q}, q \neq 0, p \leq q$ for $k \in \mathbb{Z}^+$

Consider,

$$[a_0, a_1, \ldots\ldots a_k, a_{k+1}] = \cfrac{1-\cfrac{1-\cfrac{\ddots\, 1-\frac{1}{a_{k+1}}}{a_k}}{a_2}}{a_1}\bigg/ a_0$$ k terms

$$= \cfrac{1 - \cfrac{a_1}{1 - \cfrac{a_2}{1 - \cfrac{\ddots}{\cfrac{a_k}{1 - \cfrac{1}{a_{k+1}}}}}}}{a_0}$$

$$= \frac{1-\frac{p}{q}}{a_0} = \frac{q-p}{qa_0} \epsilon \, \mathbb{Q} \quad a_0, q \neq 0.$$

By the principle of mathematical induction, the results is true for $k \, \epsilon \, \mathbb{Z}^+$.

Convergent property can be defined as follows:

Definition 1.3

The pierce continued fraction $\{a_0, a_1, \ldots\ldots a_n\}$ where k is a non-negative integer is called the n^th convergent of the simple finite pierce continued fraction $\{a_0, a_1, \ldots\ldots a_n\}$and is denoted by C_n.

Two sequences $\{[P_n\}, \{Q_n\}$ have been defined to obtain a better for the approximation to square root problem

Theorem 1.2

Let $a_0, a_1, \ldots\ldots a_k$ be finite or infinite sequence of real numbers. Define two sequences $\{[P_n\}, \{Q_n\}$ respectively as follows,

$P_0 = 1$ $\qquad$ $Q_0 = a_0$

$P_1 = a_1 - 1$ $\qquad$ $Q_1 = a_1 a_0$

.

.

$P_n = a_n P_{n-1} + (-1)^n$ $\qquad$ $Q_n = Q_{n-1} a_n$ for $n \in Z^+$

Then $C_n = \frac{P_n}{Q_n}$.

Proof

When we prove this theorem, general formula for $\{P_n\}, \{Q_n\}$have been found and then we can consider the convergent part.

Now consider $\{P_n\}$ sequence,

$P_0 = 1$

$P_1 = a_1 - 1$

.

$P_n = a_n P_{n-1} + (-1)^n$

Now clearly $P_k = a_1 a_2 \ldots\ldots a_k + (-1) a_2 \ldots\ldots a_k + (-1)^2 a_3 \ldots\ldots a_k + \ldots\ldots + (-1)^{k-1} a_k + (-1)^k$

We can prove this by P.M.I

$k = 1$

$P_1 = a_1 + (-1) = a_1 - 1$

Assume that it is true for $k = p$;

Therefore

$P_p = a_1 a_2 \ldots\ldots a_p + (-1) a_2 \ldots\ldots a_p + (-1)^2 a_3 \ldots\ldots a_p + \ldots\ldots + (-1)^{p-1} a_p + (-1)^p$ ……(1)

Then by (1) $\times \, a_{p+1}$,

we get

$$P_p a_{p+1} = a_1 a_2 \ldots\ldots a_p a_{p+1} + (-1) a_2 \ldots\ldots a_p a_{p+1} + (-1)^2 a_3 \ldots\ldots a_p a_{p+1} + \ldots\ldots + (-1)^{p-1} a_p a_{p+1} + (-1)^p a_{p+1}$$

But we know that

$$P_{p+1} = P_p a_{p+1} + (-1)^{p+1}$$
$$P_{p+1} - (-1)^{p+1} = a_{p+1} P_p$$
$$P_{p+1} = a_1 a_2 \ldots\ldots a_p a_{p+1} + (-1) a_2 \ldots\ldots a_p a_{p+1} + (-1)^2 a_3 \ldots\ldots a_p a_{p+1} + \ldots\ldots + (-1)^{p-1} a_p a_{p+1} + (-1)^p a_{p+1} + (-1)^{p+1}$$

Therefore, the result is true for $p \in \mathbb{Z}^+$.

Therefore

$P_n = a_1 a_2 \ldots\ldots a_n + (-1) a_2 \ldots\ldots a_n + (-1)^2 a_3 \ldots\ldots a_n + \ldots\ldots + (-1)^{n-1} a_n + (-1)^n$…..(2)

Similarly, for $\{Q_n\}$ sequence,

$Q_n = a_0 a_1 \ldots\ldots a_n$ ………………………………(3)

Now consider,

$\frac{(2)}{(3)}$,

$$\frac{P_n}{Q_n} = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \ldots + \frac{(-1)^{n+1}}{a_1 a_2 a_3 \ldots\ldots a_n}.$$

$$= \cfrac{1 - \cfrac{1 - \cfrac{1 - \cfrac{1^{\cdot^{\cdot^{\cdot^{.1-\frac{1}{a_3}}}}}}{a_4}}{a_3}}{a_2}}{a_1} = \{a_1, a_2, \ldots\ldots a_n\} = C_n$$

(By definition).

Using above results following identities can be easily formed,

Theorem 1.3

Let $C_n = \frac{P_n}{Q_n}$ be n^th convergent of the pierce continued fraction $\{a_0, a_1, \ldots\ldots a_n \ldots\}$ where n is positive integer. Then

i. $P_n Q_{n-1} - P_{n-1} Q_n = (-1)^n Q_{n-1}$
ii. $P_n Q_{n-2} - P_{n-2} Q_n = (-1)^{n-1}(a_n - 1) Q_{n-2}; n \geq 2$

Proof.

Proofs can be done by Principle of Mathematical Induction

Corollary 1

i. $\boldsymbol{C_n - C_{n-1} = \frac{(-1)^n}{Q_n}}$

ii. $\boldsymbol{C_n - C_{n-2} = \frac{(-1)^{n-1}(a_n - 1)}{Q_n}}$.

Proofs can be prepared using Theorem 1.4.

Example 2

Approximation value of $\sqrt{5}$ can be determined using pierce continued fraction as follows;

We have re written $\sqrt{5}$, this way

$$x = \sqrt{5} = 2 + (\sqrt{5} - 2)$$

Using above algorithm, we can construct simple finite pierce continued fraction (S.F.P.C.F) for $(\sqrt{5} - 2).(\sqrt{5} - 2) = [4,17,19,57,77,\ldots]$. Then we can construct two sequences that we have mentioned above.

$q^1{}_i$	0	1	2	3
$C^1{}_i$	4	17	19	77
P_i	1	16	305	1761984
Q_i	4	68	1292	7463884
C_i	0.25	0.2352941176	0.2360681115	0.2360679775

$$C_0 = 2.25$$
$$C_1 = 2.2352941176$$
$$C_2 = 2.2360681115$$
$$C_3 = 2.2360679775$$

Now to get final answers we have to add 2 for each C_n value. Our new C_n sequence is,

Exact value of the $\sqrt{5}$ is, $\sqrt{5} = 2.2360679977$.

By definition 1.3, n^th convergent is denoted by C_n, the actual value *x* of the infinite pierce continued fraction will lies between two strings, one string made up of odd convergent, and other made up of even convergent. Thus,

$$C_0 \geq C_2 \geq C_4 \ldots\ldots \geq x \geq \ldots\ldots \geq C_5 \geq C_3 \geq C_1$$

According to this we can obtain following theorems,

Theorem 1.4

Let $x = [a_0, a_1, a_2, \ldots\ldots a_n \ldots]$, where $C_n$is n^th convergent. Then,

i. $\lim_{n\to\infty} C_n = x$. Further $x \gtreqless C_n$ according to n is odd or even.

ii. $|x - C_n| \geq |x - C_{n+1}|$

iii. $|x - C_n| < \frac{1}{q_{n+1}}$

Proof

i. Since $C_1 \leq C_3 \leq C_5 \leq \ldots\ldots$is increasing sequence. Bounded above by any even convergent.

$\therefore \lim_{n\to\infty} C_{2n-1}$ exist.

$$C_1 \leq C_3 \leq C_5 \leq C_7 \leq \lim_{n\to\infty} C_{2n}$$

Similarly, $C_0 \geq C_2 \geq C_4 \ldots\ldots\ldots$ is an decreasing sequence which is bounded below by any odd convergent.

$\therefore \lim_{n\to\infty} C_{2n}$ exist.

$$C_0 \geq C_2 \geq C_4 \ldots\ldots \geq \lim_{n\to\infty} C_{2n-1}$$

Since $Q_n = Q_{n-1}a_n : \{Q_n\}$is an increasing sequence

Therefore $Q_n \to \infty$as $n \to \infty$

Since $\boldsymbol{C_n - C_{n-1} = \frac{(-1)^n}{Q_n}}$ by corollary 1 part (iv)

Therefore $\lim_{n\to\infty} C_{2n} = \lim_{n\to\infty} C_{2n-1}$

$$\lim_{n\to\infty} C_{2n} = \lim_{n\to\infty} C_{2n-1} = \lim_{n\to\infty} C_{2n}$$

i.e. $x = [a_0, a_1, a_2, \ldots\ldots a_n, a_{n+1} \ldots]$

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \ldots + \frac{(-1)^{n+1}}{a_1 a_2 a_3 \ldots a_n} + \frac{(-1)^{n+2}}{a_1 a_2 a_3 \ldots a_{n+2}}$$

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \ldots + \frac{(-1)^n}{a_1 a_2 a_3 \ldots\ldots a_n} + \frac{(-1)^{n+1}}{\underbrace{\boxed{a_1 a_2 a_3 \ldots\ldots a_n}}_{Q_n}} \underbrace{\left\{ \cdot \frac{1}{a_{n+1}} + \frac{(-1)^1}{a_{n+1} a_{n+2}} \right\}}_{y}$$

This can be written as,

$x = C_n + \frac{y(-1)^{n+1}}{Q_n}$……………………………….(4)

Now consider,

$|x - C_n| = \left|\frac{y(-1)^{n+1}}{Q_n}\right| = \frac{y}{Q_n}$ but we know $\frac{y}{Q_n} \geq 0$

But we know $y < 1$

Then,

$\frac{y}{Q_n} < \frac{1}{Q_n}$,$0 \leq |x - C_n| < \frac{1}{Q_n}$ ……………………(5)

But $n \to \infty, \frac{1}{Q_n} \to 0 (\because Q_n > 0)$

By sandwich lemma with (5)

$$\lim_{n\to\infty} C_n = x$$

Therefore,

$C_0 \geq C_2 \geq C_4 \ldots\ldots \geq x \geq \ldots\ldots \geq C_5 \geq C_3 \geq C_1$.

i.e. $x \gtreqless C_n$ according to n is odd or even.

ii. By (4), $x = C_n + \frac{y(-1)^{n+1}}{Q_n}$

$$|x - C_n| = \left|\frac{y}{Q_n}\right| = \frac{y}{Q_n} ------(A)$$

$x = C_{n+1} + \frac{y(-1)^{n+2}}{Q_{n+1}}$………………………(6)

$$|x - C_{n+1}| = \left|\frac{y(-1)^{n+2}}{Q_{n+1}}\right| = \frac{y}{Q_{n+1}} ------(B), n \geq 1$$

We know Q_n is an increasing sequence

$$\therefore Q_n \leq Q_{n+1} \Rightarrow \frac{1}{Q_n} \geq \frac{1}{Q_{n+1}} \Rightarrow \frac{y}{Q_n} \geq \frac{y}{Q_{n+1}}.$$

Therefore $|x - C_n| \geq |x - C_{n+1}|$.

iii. By Corollary 1 part (iv)

$$\boldsymbol{C_n - C_{n-1} = \frac{(-1)^n}{Q_n}}$$

$|C_n - C_{n-1}| = \left|\frac{(-1)^n}{Q_n}\right| = \frac{1}{Q_n}$……………(7)

By above part ii., x lies between $C_n$and C_{n+1}

$$|x - C_n| < |C_{n+1} - C_n| = \frac{1}{Q_{n+1}}$$

$$|x - C_n| < \frac{1}{Q_{n+1}}$$

Last part has been given for upper bound for given approximation square root problem.

II. RESULTS AND DISCUSSION

Using our proposed algorithm, we establish couple of results.

Result 1. For any real number $x\,(0 < x < 1)$, suppose that the Pierce contined fraction of $x$is$\langle a, b, a, b, .. \rangle$ (for example, $\frac{3}{11} = \langle 3,4,3,4, \dots \rangle$). This infinite Pierce continued fraction can be written as a finite continued fraction,

$x = \langle a, b, a, b, .. \rangle = \langle a, b, ab - 1 \rangle$, where $b = a + 1$.

Result 2. For any real number $x, (0 < x < 1)$, suppose that the Pierce continued fraction of x is $\langle a, b \rangle$ (for example, $\frac{12}{117} = \langle 9,13 \rangle$). This finite continued fraction can be written as an infinite continued fraction,

$$x = \langle a, b \rangle = \langle a, b - 1, b - 1, b - 1, \dots \rangle$$

In previous theorem last part have been discussed an upper bound for square root problem, for an example considering the value of $\sqrt{5}$ correct to four decimal places,

If $|\sqrt{5} - C_n| < 0.00005$ then $C_n - 0.00005 < \sqrt{5} < C_n + 0.00005$and $C_n$gives the value of $\sqrt{5}$ correct to four decimals places.by theorem 1.3 part (iii), $|x - C_n| < \frac{1}{Q_{n+1}}$

If $\frac{1}{Q_{n+1}} < 0.00005$then$|\sqrt{5} - C_n| < 0.00005$.

i.e., $Q_n > 20{,}000$.

According example 2,

Using continued fraction approximation value of $\sqrt{5}$in to 4 decimal places is given by the 6th convergent (c_6)

But using the pierce continued fraction same result can be obtained by 4th convergent or less.

In addition, we have illustrated above results graphically. Table 1, we get this value when n=3. But if we used standard continued fraction minimum value for n is 6.

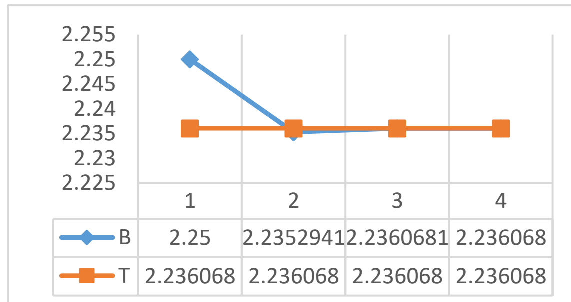


Fig. 1- exact value of $\sqrt{5}$ vs C_n

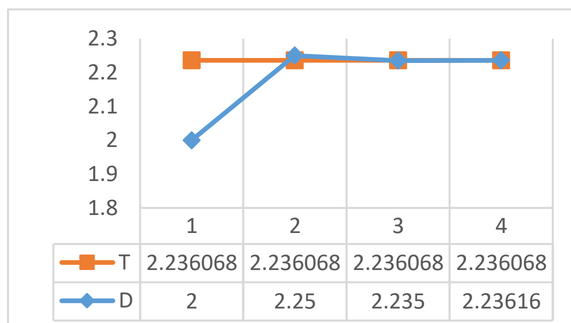


Fig. 2- exact value of $\sqrt{5}$ vs c_n

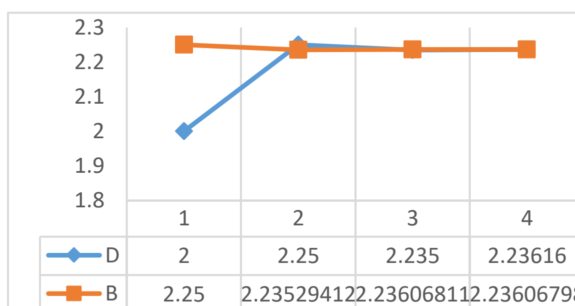


Fig. 3- $C^1{}_n$ vs c_n

In figure 1, graph of C_n has been coincide with graph of $\sqrt{5}$ subsequently $n = 3$. According to the above graphs, the variance of the partial quotients of pierce continued fraction are smaller than partial quotients of usual continued fraction.

CONCLUSION

In this work we introduce a continued fraction called the Pierce continued fraction which has a better convergent rate than the usual continued fraction. Using this advantage, we got appropriate approach for approximation to square root problem. Moreover, two sequences $\{P_n\}$ and $\{Q_n\}$ are defined to obtain the relevant convergence. In addition, we have proved theorem which are necessary to obtain the convergence of a pierce continued fraction. Most of the proofs are similar to the proofs of their continued fraction.

There are lots of applications related to continued fractions. Solving linear Diophantine equations is one of the major applications. As a future work, we expect to establish a result to solve the Pelle's equation using Pierce continued fraction.

REFERENCES

[1] Cor. Kraaikamp and Jun Wu, On a New Continued Fraction Expansion with Non-Decreasing Partial Quotients (2004).

[2] Peter J. Cameron A Course on Number Theory, Queen Mary, University of London (2009).

[3] William Stein Elementary Number Theory: Primes, Congruence's, and Secrets. (2011).

[4] Rotman, Joseph, *An introduction to the theory of groups, New York: Springer-Verlag, ISBN 0-387-94285-8* A standard contemporary reference, (1994)