On class (BQ) Operators of order n.

WANJALA VICTOR[1], OYOMBE ALUALA[2]
[1, 2] *Department of Mathematics and computing, Kibabii University.*

Abstract- In this paper, we extend the class of (BQ) operators acting on a complex Hilbert space H to the class of (BQ) of order n. An operator if T ∈ B(H) is said to belong to class (BQ) of order n if $T^{*2n}T^2$ commutes with $(T^{*n}T)^2$ that is $[T^{*2n}T^2, (T^{*n}T)^2] = 0$. We examine properties that this class is honored to have. We examine the relation of this class to that of class (Q) order n.

Indexed Terms- Class (BQ) of order n, Class (BQ) operator, Normal operator of order n.

I. INTRODUCTION

Throughout this paper , H denotes the usual Hilbert space over the complex field and B(H) the Banach algebra of all bounded linear algebra on an infinite dimensional separable Hilbert space H . An operator T ∈ B(H) is said to be class (Q) if $T^{*2}T^2 = (T^*T)^2$ (1), class (Q) of order n if $T^{*2n}T^2 = (T^{*n}T)^2$, class (BQ) if $T^{*2}T^2(T^*T)^2 = (T^*T)^2T^{*2}T^2$ (5). A conjugation on a Hilbert space H is an anti-linear operator C from Hilbert space H onto itself that satisfies

Cx, Cyi = hx, yi for every x, y ∈ H and $C^2 = I$. An operator T is said to be complex symmetric if $T = CT^*C$.

II. MAIN RESULTS

Theorem 1. Let T ∈ B(H) be such that it's a (BQ) of order n, then the following are also equivalent;
(i). λT for any real λ
(ii). Any S ∈ B(H) that is unitarily equivalent to T.
(iii). The restriction T-M to any closed subspace M of H.

Proof. (i). The proof is trivial.
(ii). Let S ∈ B(H) be unitarily equivalent to T, then there exists a unitary operator U ∈ B(H) with $S = U^*TU$ and $S^* = U^*T^{*n}U$. Since T ∈ B(Q) of order n , we have; $T^{*2n}T^2(T^{*n}T)^2 = (T^{*n}T)^2T^{*2n}T^2$, hence
$S^{*2n}S^2(S^{*n}S)^2 = UT^{*2n}U^*UT^2U^*(UT^{*n}U^*UTU^*)^2$
$= UT^{*2n}U^*U^*T^2U^*UT^{*n}U^*UT^{*n}U^*UTU^*UTU^*$
$= UT^{*2n}T^2(T^{*n}T)^2U^*$
$= U(T^{*n}T)^2T^{*2n}T^2U^*$
and
$(S^{*n}S)^2S^{*2n}S^2 = (UT^{*n}U^*UTU^*)^2UT^{*2n}U^*UT^2U^*$
$= UT^{*n}U^*UTU^*UT^{*n}U*UTU^*UT^{*2n}U^*UT^2U^*$
$= UT^{*n}TT^{*n}TT^{*2n}T^{*2n}U^*$
$= U(T^{*n}T)^2T^{*2n}T^2U^*$
Thus S is unitarily equivalent to T.

(iii) . If T is in class (BQ) of order n , then; $T^{*2n}T^2(T^{*n}T)^2 = (T^{*n}T)^2T^{*2n}T^2$. Hence;
$(T/M)^{*2n}(T/M)^2\{(T/M)^{*n}(T/M)\}^2$
$=(T/M)^2(T/M)^2\{(T/M)^{*n}(T/M)\}^2$
$= (T^{*2n}/M)(T^2/M)\{(T^{*n}/M)(T/M)\}\{(T^{*n}/M)(T/M)\}$
$= \{(T^{*n}T)^2/M\}\{T^{*2n}T^2/M\}$
$= \{(T^{*n}/M)(T/M)\}^2(T/M)^{*2n}(T/M)^2$
Thus T/M ∈ (BQ) of order n .

Theorem 2. If T ∈ B(H) is in Class (Q) of order n, then T ∈ (BQ) of order n .

Proof. If T ∈ (Q) of order n, then $T^{*2n}T^2 = (T^{*n}T)^2$ post multiplying both sides by $T^{*2n}T^2$;
$T^{*2n}T^2T^{*2n}T^2 = (T^{*n}T)^2T^{*2n}T^2$
$T^{*2n}T^2T^{*n}TT^{*n}T = (T^{*n}T)^2T^{*2n}T^2$
$T^{*2n}T^2(T^{*n}T)^2 = (T^{*n}T)^2T^{*2n}T^2$.

Theorem 3. Let S, T ∈ (BQ) of order n. If both S and T are doubly commuting, then ST is in (BQ) of order n.

Proof.
$(ST)^{*2n}(ST)^2((ST)^{*n}(ST))^2$
$=S^{*2n}T^{*2n}S^2T^2((ST)^{*n}(ST))((ST)^{*n}(ST))$
$=S^{*2n}T^{*2n}S^2T^2((S^{*n}T^{*n})(ST))((S^{*n}T^{*n})(ST))$
$=S^{*2n}T^{*2n}S^2T^2S^{*n}T^{*n}STS^{*n}T^{*n}STS^{*n}T^{*n}ST$
$= S^{*2n}T^{*2n}S^2T^2S^{*n}ST^{*n}TS^{*n}ST^{*n}T$
$=T^{*2n}T^2S^{*2n}S^2S^{*n}SS^{*n}ST^{*n}TT^{*n}T$

$=T^{*2n}T^2S^{*2n}S^2(S^{*n}S)^2T^{*n}TT^{*n}T$

$=T^{*2n} T^2 (S^{*n}S)^2 S^{*2n} S^2 T^{*n} TT^{*n}T$ (Since $S \in (BQ)$ of order n).
$= (S^{*n}S)^2 T^{*2n} T^2 T^{*n} TT^{*n}TS^{*2n} S^2$
$=(S^{*n}S)^2 T^{*2n} T^2 (T^{*n}T)^2 S^{*2n} S^2$
$=(S^{*n}S)^2 (T^{*n}T)^2 T^{*2n} T^2 S^{*2n} S^2$ (Since $T \in (BQ)$ of order n).
$=((S^{*n}S)(T^{*n}T))^2 T^{*2n} S^{*2n} T^2 S^2$
$=((S^{*n}T^{*n})(ST))^2 S^{*2n} T^{*2n} S^2 T^2$
$= ((ST)^{*n}(ST))^2 (ST)^{*2n} (ST)^2$
Thus $ST \in (BQ)$ of order n .
Theorem 4. Let $T \in B(H)$ be a class (BQ) operator of order n such that $T = CT^{*n}C$ with C being a conjugation on H. If C is such that it commutes with $T^{*2n} T^2$ and $(T^{*n}T)^2$, then T is a class (Q) operator of order n.
Proof. Let $T \in (BQ)$ of order n and complex symmetric, then we have; $T^{*2n} T^2 (T^{*n}T)^2 = (T^{*n}T)^2 T^{*2n}T^2$ and $T = CT^{*n}C$. hence;
$T^{*2n} T^2(T^{*n}T)^2 = (T^{*n}T)^2 T^{*2n} T^2$
$T^{*2n}T^2 CTCCT^{*n} CCTCCT^{*n} C = (T^{*n}T)^2 CTCCT^{*n} CCTCCT^{*n} C.$
$T^{*2n} T^2 CTT^{*n} TT^{*n} C = (T^{*n}T)^2 CTT^{*n}TT^{*n} C$
$T^{*2n} T^2 CT^2 T^{*2n} C = (T^{*n}T)^2 CT^{*n}TT^{*n}TC$
$T^{*2n}T^2 CT^{*2n} T^2 C = (T^{*n}T)^2 C(T^{*n}T)^2 C.$
C commutes with $T^{*2} T^2$ and $(T^{*}T)^2$ thus we get ;
$T^{*2n}T^2T^{*2n}T^2 = (T^{*n}T)^2 (T^{*n}T)^2$. which implies ;
$T^{*2n}T^2 = (T^{*n}T)^2$ and hence $T \in (Q)$ of order n.

REFERENCES

[1] Jibril, A.A.S.., On Operators for which $T^{*2}(T)^2 = (T^*T)^2$, international mathematical forum, vol. 5(46) ,2255-2262.

[2] S. Paramesh, D. Hemalatha and V.J. Nirmala., A study on n-power class (Q) operators, international research journal of engineering and technology, vol.6(1), (2019) , 2395-0056.

[3] wanjala Victor and A.M. Nyongesa., On (α, β)-class (Q) Operators, international journal of mathematics and its applications, vol. 9(2) (2021), 111-113.

[4] Wanjala Victor and Beatrice Adhiambo Obiero., On almost class (Q) and class (M,n) operators ,international journal of mathematics and its applications ,vol . 9(2) (2021), 115-118.

[5] Wanjala Victor and Beatrice Adhiambo Obiero., On class (BQ) operators, Global Journal of Advanced Research, vol. 8(4) (2021), 118-120.