Maximal Subgroups of Some Groups of Extension

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***Abstract-* Let G be a finite group. A maximal subgroup H of a group G is a proper subgroup, such that no proper subgroup K of G strictly contains H. If N and G are groups, an extensionof N by G is a group M such that N⊴M and M/N≅G. In this paper, we determinegroups of extension $O_8^+(2):2$, $L_3(4):2$, $L_3(4):2^2$ and $L_3(3)$: 2 from some finitegroups using modular representation method. We determine the maximal subgroups from the group extensions. We determine the degree, order, number of orbits and the length of the orbits to classify the maximal subgroups obtained from groups of extension.**

I. INTRODUCTION

Definition 1.1. *A group is a set G together with a binary operation ∗such that:*
(*i*). $(a * b) * c = a * (b * c)$ *for all a, b, c∈G*
(*ii*). $e * a = a * e = a$, *for all a ∈G and e is the identity element in G.*
(*iii*). $a * a^{-1} = a^{-1} * a = 1$, *for all a ∈G where* a^{-1} *is the inverse of a* [1].
Definition 1.2. *A maximal subgroup H of a group G is a proper subgroup, such that no proper subgroup K of G strictly contains H* [2].
Definition 1.3. *When a group G acts on a set X, the orbit of an element x is the set* $G(x) = \{g.x \in X : g \in G\}$ [3].

Definition 1.4. *If N and G are groups, an extension of N by G is a group M such that:*

(*i*). $N \trianglelefteq M$

(*ii*). $M/N \cong G$
We denote the fact that *M is* an extension of N by G by writing $M = N \cdot G$ [4].
Definition 1.5. *If M is an extension of N by G such that there exists a subgroup* G_1 *of M and:*
(*i*). $G_1 \cong G$
(*ii*). $N \cdot G_1 = M$
(*iii*). $N \cap G_1 = 1$.
When all the three properties are satisfied, then we say that *M*is a split extension andwe denote this fact by writing *M* = N: G. If an extension is not split, then it is called non-split and is denoted by N.G [4].
Let F be a field of characteristic *ρ*, and let *V* be an F vector space. Let *G* be a finite group of order n.
Definition 1.6. *A linear representation of G over* F *is a homomorphism* $\rho : G \to GL(V)$[5].
Definition 1.7. *A vector space V over the field* F *is an* F*G-module if there exists a multiplication* $G \times V \to V$, *satisfying:*
(*i*). $gv \in V$
(*ii*). $(gh)v = g(hv)$
(*iii*). $1v = v$
(*iv*). $g(\lambda v) = \lambda gv$
(*v*). $g(v + w) = gv + gw$ [2]
A representation inherits any of the above properties.

2. Maximal subgroups of $O_8^+(2):2$

$O_8^+(2)$: 2 is a group extension of $S_6(2)$, such that $S_6(2)$ is a normal subgroup. Table 1 shows maximal subgroups of $O_8^+(2)$: 2 and their properties.

Maximal subgroup	Degree	Order	No.of Orbits	Length of orbit
$S_6(2):2$	120	2,903,040	3	[1 , 56,63]
26 : *S*8	135	2,580,480	2	[56,64]
S_9	960	362,880	2	[36,84]
$2^2 : S_4(3) : 3$	1120	311,040	3	[3 , 36,81]
213 : 33	1575	221,184	2	[24,96]
$L_2(7) : 2^{10}$	2025	172,032	2	[8,112]
27 : 35	11200	31,104	2	[12, 108]
$2:S_5 : S_5$	12,096	28,800	2	[56,64]

Table 1: Maximal subgroups of $O_8^+(2):2$

$O_8^+(2)$: 2 has 8 maximal subgroups. They have been arranged according to the size of their degrees. The first maximal subgroup, $S_6(2)$: 2 of degree 120 with order 2,903,040 has 3 orbits of length 1, 56 and 63 respectively. The second maximal subgroup $2^6 : S_8$ of degree 135 with order 2,580,480 has 2 orbits of length 56 and 64. The third maximal subgroup S_9 of degree 960 with order 362,880 has 2 orbits of length 36 and 84 respectively. The fourth maximal subgroup $2^2 : S_4(3)$: 3 of degree 1120 with order 311,040 has 3 orbits of length 3, 36 and 81 respectively. The fifth maximal subgroup $2^{13} : 3^3$ of degree 1575 with order 221,184 has 2 orbits of length 24 and 96 respectively. The sixth maximal subgroup $L_2(7)$: 2^{10} of degree 2025 with order 172,032 has 2 orbits of length 8 and 112 respectively. The seventh maximal subgroup $2^7 : 3^5$ of degree 11200 with order 31,104 has 2 orbits of length 12 and 108. The eighth subgroup $2{:}S_{5:}S_5$ of degree 12,096 with order 28,800 has 2 orbits of length 56 and 64 respectively.

3. Maximal subgroups of$L_3(4)$:2

$L_3(4)$: 2 is a group extension of $L_3(4)$ such that $L_3(4)$ is a normal subgroup. Table 2 shows maximal subgroups of $L_3(4)$: 2 and their properties.

Maximal subgroup	Degree	Order	Orbits	Length of orbits
$2 : A_5 : 2^4$	21	1920	2	[40 ,80]
$2 : A_5 : 2^4$	21	1920	2	[40 ,80]
$2 : A_6$	56	720	3	[15, 15 , 90]
$L_2(7) : 2$	120	336	4	[1, 21, 42, 56]
24× 32	280	144	4	[12,18, 18, 72]

Table 2: Maximal subgroups of $L_3(4)$: 2

$L_3(4)$: 2 has 5 maximal subgroups. The first maximal subgroup $2 : A_5 : 2^4$ of degree 21 with order 1920 has 2 orbits of length 40 and 80 respectively.The second maximal subgroup $2 : A_5 : 2^4$ of degree 21 with order 1920 has 2 orbits of length 40 and 80 respectively. The third maximal subgroup $2 : A_6$ of degree 56 with order 720 has 3 orbits of length 15, 15 and 90 respectively. The fourth maximal subgroup $L_2(7)$: 2 of degree 120 with order 336 has 4 orbits whose lengths are 1, 21, 42 and 56 respectively. The fifth maximal subgroup $2^4 \times 3^2$ of degree 280 with order 144 has 4 orbits of length 12, 18, 18 and 72 respectively.

4. Maximal subgroups of $L_3(4)$: 2^2

$L_3(4)$: 2^2 is a group extension of $L_3(4)$. Table 3 showsmaximal subgroups of $L_3(4)$: 2^2 and their properties.

Maximal subgroup	Degree	Order	Orbits	Length of orbits
$A6 : 22$	56	1440	2	[30 ,90]
$2^8 \times 3$	105	768	3	[24 ,32, 64]
$L_2(7) : 2^2$	120	672	4	[1, 21, 42, 56]
25× 32	280	288	3	[12, 36, 72]
$22 : A5$	336	240	4	[10,20, 30, 60]

Table 3: Maximal subgroups of $L_3(4)$: 2^2

$L_3(4)$: 2^2 has five maximal subgroups. The first maximal subgroup $A_6 : 2^2$ of degree 56 with order 1440 has 2 orbits with length 30 and 90 respectively. The second maximal subgroup $2^8 \times 3$ of degree 105 with order 768 has 3 orbits of length are 24, 32 and 64 respectively. The third maximal subgroup $L_2(7)$: 2^2 of degree 120 with order 672 has 4 orbits of length 1, 21, 42 and 56 respectively. The fourth maximal subgroup $2^5 \times 3^2$ of degree 280 with order 288 has 3 orbits of length 12, 36 and 72 respectively. The fifth maximal subgroup $2^2 : A_5$ of degree 336 with order 240 has 4 orbits of length are 10, 20,30 and 60 respectively.

5. Maximal subgroups of $L_3(3)$:2

$L_3(3)$: 2 is a group extension of $L_3(3)$ such that $L_3(3)$ is a normal subgroup. Table 4 shows maximal subgroups of $L_3(3)$: 2 and their properties.

Maximal subgroup	Degree	Order	Orbits	Length of orbits
23× 33	52	216	2	[36 ,108]
$2^5 \times 3$	117	96	3	[48, 48, 48]
$2 \times 3 \times 13$	144	78	6	[1, 13, 26, 26, 39, 39]
$2^4 \times 3$	234	48	6	[8, 16, 24, 24, 24, 48]

Table 4: Maximal subgroups of $L_3(3)$: 2

$L_3(3)$: 2 has four maximal subgroups. The first maximal subgroup $2^3 \times 3^3$ of degree 52 with order 216 has 2 orbits of length 36 and 108 respectively. The second maximal subgroup $2^5{\times}3$ of degree 117 with order 96 has 3 orbits of length 48, 48 and 48 respectively. The third maximal subgroup $2\times3\times13$ of degree 144 with order 78 has 6 orbits of length 1, 13, 26, 26, 39 and 39 respectively. The fourth maximal subgroup $2^4 \times 3$ of degree 234 with order 48 has 6 orbits of length 8, 16, 24, 24, 24 and 48 respectively.

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