k - Odd Prime Labeling of m×n Grid Graphs

W. K. M. INDUNIL[1], K. N. KALUARACHCHI[2], A. C. G. PERERA[3]

[1, 2] *Faculty of Applied Sciences, Department of Physical Science, Rajarata University of Sri Lanka, Sri Lanka.*

[3] *Department of Mechanical Engineering,Open University of Sri Lanka, Nawala, Nugegoda, Sri Lanka.*

Abstract—Graph labeling can be mentioned as one of the most prominent research areas in graph theory andthe history of graph labeling can be traced back to the 1960s as well. There is aquite number of graph labeling techniques such as graceful labeling, radio labeling, antimagic labeling, prime labeling, and lucky labeling. There are varioussubtypes of prime labeling including odd prime labeling, k- prime labeling, neighborhood prime labeling, and coprime labeling. In this study, we explore one of the prime labeling varieties called odd prime labeling. There is a well-known conjecture related to this area that all the prime graphs are odd prime graphs. There is a vast number of publications regarding prime labeling and odd prime labeling fordifferent classes of graphs. Recent works on odd prime labeling investigate different types of snake graphs, complete graphs, triangular-type snake graphs, different types of ladder graphs, families of cycle-related and path-related graphs, etc. In this research work, we introducethe concept of k- odd prime labeling and obtain several k- odd prime graphs such asm×ngrid graph and variations of it.

Indexed Terms—m×ngrid graph, ladder graph,k- odd prime graph,k- odd prime labeling

I. INTRODUCTION

The concept of prime labeling was conceived by Roger Entringer around the 1980s where he stated the conjecture that every tree is prime, which remains unsolved[1]. With time many variations of prime labeling such as neighborhood prime labeling, coprime labeling, k – prime labeling, and odd prime labeling were developed[2].

In this work, we mainly focused on a special type of graph labeling method called odd prime labeling which was introduced by Prajapati and Shah [3] on a simple, undirected, and connected graph.Odd prime labeling of graphs is an interesting research area with a considerable amount of literature on different types of graphs. In odd prime labeling of a simple graph, vertices are labeled with distinct odd integers from the set $\{1,3,5,7\cdots,2n-1\}$ in such a way that the labels of any two adjacent vertices are relatively prime. A graph that admits odd prime labeling is called an odd prime graph.

In 2019, U. M. Prajapati and K. P. Shah published an article that discusses odd prime labeling on various snake graphs like $n-$ polygonal snake, double $n-$ polygonal snake, alternate $n-$ polygonal snake, double alternate triangular snake, irregular triangular snake, irregular quadrilateral snake and further he mentions about as future work on the odd prime labeling of irregular snake graphs [4]. Maged and Zainab introduced some amazing properties of odd prime graphs in a paper they published in 2020 and further they discussed odd prime labeling of some special graphs and disjoint union of graphs as well [1]. As well as in 2022, H. Carter and N. Bradley discuss odd prime labeling of another type of snake graph and some cycle-related graphs [2]. Odd prime labeling of different types of subdivision graphs of quadrilateral snake, triangular type snake, etc and some ladder graphs have been discussed by G. Gajalakshmi and S. Meena in 2022 [3].

In our research work, we obtain several results regardingk-odd prime labeling of grid m×ngraph and some variations of it.

First, we begin by giving some important and useful notations and the definitions used in this research work.

Definition 1. k - odd prime labeling.

A graph G with vertex set V(G) is said to have *k* -odd prime labeling if there exists an injectivefunction f: V(G) → {k, k+2, k+4, . . . , | V(G) |}where k is an odd integer such that for every edge xy ∈ E(G).f (x) and f (y) are relativelyprime. A graph that admits k-odd prime labeling is a k-odd prime graph.

Definition2. Ladder Graph.

If P_n denote the path on n vertices, then the Cartesian product $P_m \times P_n$, where $m \leq n$, is called a grid graph. If m=2, then the graph is called a ladder($m,n \in Z^+$)[5].

II. METHODOLOGY

A. Theorem1

The m×n grid graph is k-odd prime whenever $n=2^x$ where m,n>1, x≥0, k-odd.

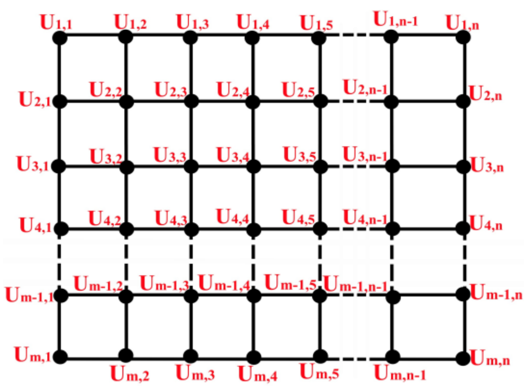

Figure 1: m × nGrid graph

Let the k-odd prime labeling function of this m×n grid graph be f.

Start the labeling from the vertex $U_{1,1}$, $f(U_{1,1})=k$, then label the vertices of the m=1 row using consecutive odd integers up until $U_{1,n}$, then move to the $U_{2,1}$ and go through the m=2 row up until $U_{2,n}$. By repeating this procedure, we can label this kind of grid graph k-odd prime.

Proof:

Since labels of any two adjacent vertices in any row have consecutive odd integers, they are relatively prime but when considering labels of any two adjacent vertices in a particular column, since they are not consecutive odd integers, they need to satisfy the following condition.

Considering the greatest common device of labels of two adjacent vertices in the first column,

$$gcd(2n(m-1)+k,2n(m-2)+k)=1$$
$$gcd(2n,2n(m-2)+k)=1$$
$$gcd(2n,k)=1 \quad (1)$$

So, labels of any adjacent two vertices are relatively prime for any m since any integer is relatively prime with 1.

Considering the greatest common device of labels of two adjacent vertices in the second column,

$$gcd(2n(m-1)+k+2, 2n(m-2)+k+2)=1$$
$$gcd(2n,2n(m-2)+k+2)=1$$
$$gcd(2n,k+2)=1 \quad (2)$$

Considering the greatest common device of labels of two adjacent vertices in the third column,

$$gcd(2n(m-1)+k+4,2n(m-2)+k+4)=1$$
$$gcd(2n,2n(m-2)+k+4)=1$$
$$gcd(2n,k+4)=1 \quad (3)$$

⋮

By continuing this procedure for each column until $(n-1)^{th}$ column,

$$gcd(2mn+k-4,2n(m-1)+k-4)=1$$
$$gcd(2n,2n(m-1)+k-4)=1$$
$$gcd(2n,2n+k-4)=1 \quad (n-1)$$

For the n^{th} column,

$$gcd(2mn+k-2,2n(m-1)+k-2)=1$$
$$gcd(2n,2n(m-1)+k-2)=1$$
$$gcd(2n,2n+k-2)=1 \quad (n)$$

So, considering equations (1) to (n), we can conclude that this type of m×n grid graph can be labeled k-odd prime if gcd(2n,2i+k-2)=1; k – odd where i is the corresponding column index.

The allowed values for 2n which gcd(2n,2i+k-2)=1; i-1,2,3,...n; k - odd can be obtained as follows.

Since the summation of an odd integer and an even integer gives an odd integer always, k+2i-2=k+2(i-1) is an odd integer as k is always odd and 2(i-1) is always even for any i. Therefore, as far as 2n is not an integer multiple of an odd integer, gcd(2n,2i+k-2)=1.

Since 2n must not be an integer multiple of an odd integer, this 2n can be only written as the multiplication of factors of even numbers only. Since any even number can be written as a multiplication of 2 and since this 2n can be only written as the multiplication of factors of even numbers, the whole

multiplication can be written as a multiplication of terms of 2's only as follows,
$2n=2\times2\times2\times2\times...\times2\times2=2^t$ where $t\geq1$. Finally, in order to label the m×n grid graph k-odd prime, $2n=2^t$; $t\geq1$. Therefore, $n=2^{t-1}=2^x$; $x\geq0$ where $x=t-1$.
So, any m×n grid graph can be labeled k-odd prime whenever $n=2^x$; $x\geq0$; k - odd.

B. Remark 1

Disjoint union of any finite number of $m_i\times n_i$ grid graph is k-odd prime whenever in each grid graph $n_i=2^x$; $x\geq0$; k - odd; i - 1, 2, 3, ...

C. Remark 2

An irregular chain of $m_i\times n_i$ grid graph is k - odd prime whenever in each grid graph $n_i=2^x$; $x\geq0$; k - odd; i - 1, 2, 3, ...
This graph can be constituted by connecting $m_i\times n_i$ grid graphs each other using path graphs, P_{l_j} or directly gluing $U^i_{m_i,n_i}$ and $U^{i+1}_{1,1}$ where i-1, 2, ...,L-1 and j∈N.
Refer to Fig. 4.
Proof:
In this scenario also it is quite obvious that the first grid graph can be labeled using Theorem 1 and the

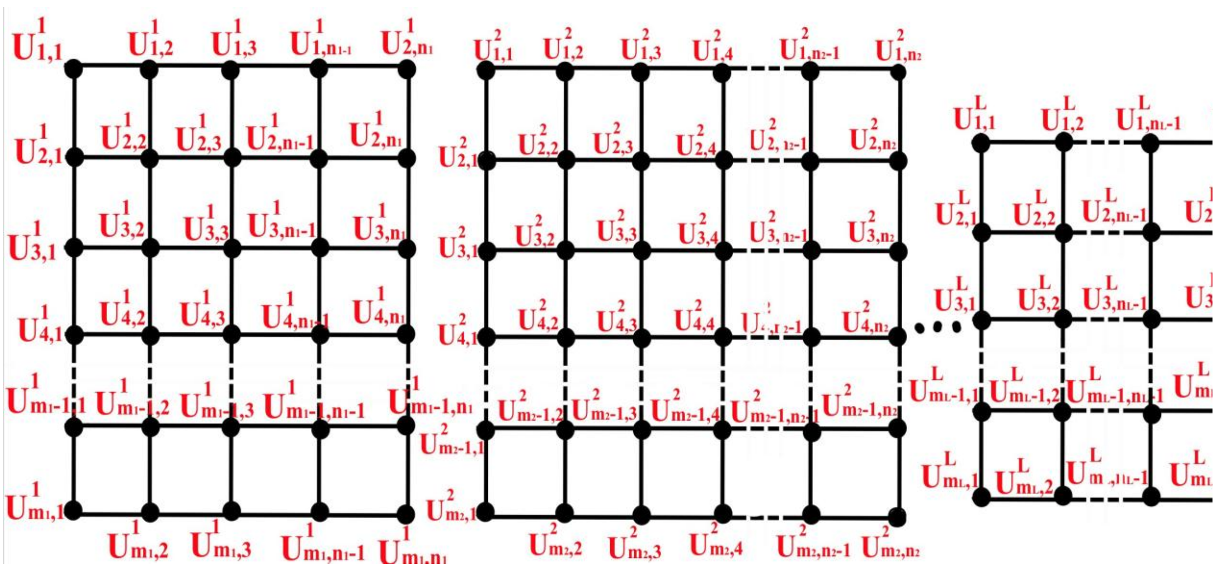

Figure 2: Disjoint union of grid graphs

Refer to Fig. 2.
After labeling the first grid, L=1, as same as the grid in Fig. 1, we start the labeling from the vertex $U^2_{1,1}$ taking the $k=2m_1n_1+1$ for L=2. Then continue as usual.
Proof:
It is obvious that the first grid graph can be labeled using Theorem 1 and the first label of the next grid graph will be the $2(m_1n_1)+1$. Since $2(m_1n_1)+1$ is an odd integer, by using Theorem 1 again for $k=2(m_1n_1)+1$, we can label the 2nd grid graph odd prime. By repeating Theorem 1 for different k values which depend on the last label of the previous grid graph, we can label any finite number of grid graphs k-odd prime.

first label of the next path graph, P_{l_1} will be $2(m_1n_1)+1$. Since this is a path graph, we can label it by using consecutive odd integers, and the last label of the P_{l_1} will be $2(m_1n_1+l_1)-1$, which is also an odd integer, so the next grid graph can be labeled odd prime by using the Theorem 1 setting as the $k=2(m_1n_1+l_1)+1$. By repeating this procedure for different k values which depend on the last label of the previous path graph, we can label the chain of a finite number of grid graphs odd prime.
When $n=2^1=2$, the k - odd prime labeling of the disjoint union of ladder graphs, and the chain of ladder graphs can be obtained for any finite length of m
By substituting n=2 and k=1, we can label the ladder graph, $P_2\times P_m=L_m$ odd prime for any finite length of

m using Theorem 1, surprisingly we get the same labels on each vertex for the ladder graph when the same ladder graph is labeled using the already proven results of the research work carried out by Prajapati and Shah.

Further, we discuss thek - odd prime labeling for roach graphs as well.

D. Remark 3

All roach graphs, $R_{2(p,q)}$ are k - odd prime.

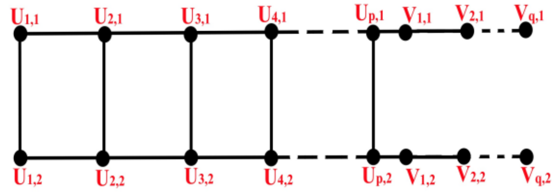

Figure 3: Roach graph

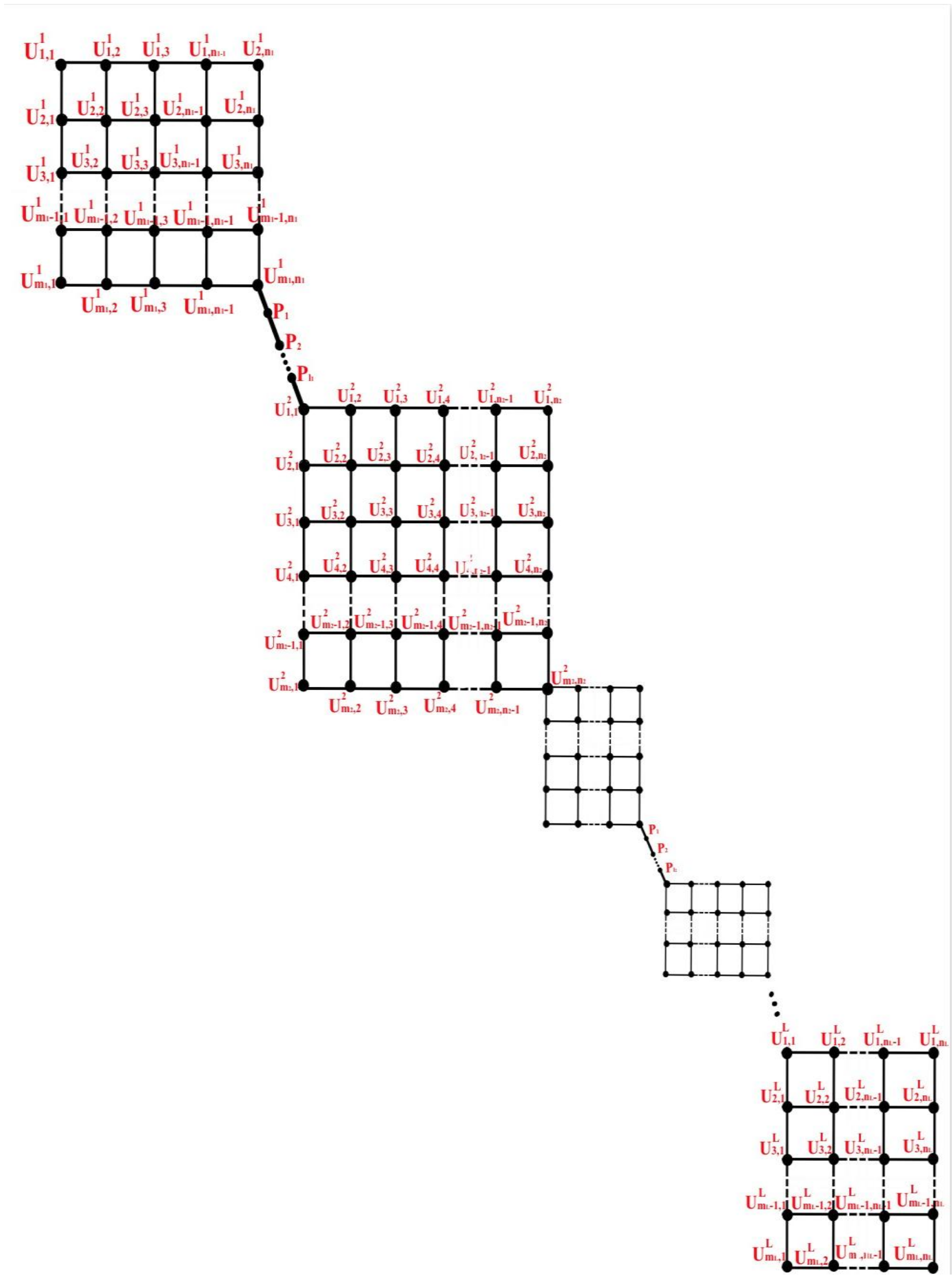

Figure 4: Irregular chain of $m_i \times n_i$ grid graph

Proof:

Since all the ladder graphs, L_m are k - odd prime, by removing the p number of vertical lines from one of the ends of the ladder graph and labeling it as the ladder graph without caring about the structure using Theorem 1, we can have the k - odd prime labeling where m=p+q. Since we only remove vertical edges from the ladder graph to create the roach graph and it does not create any, we don't need to check the relatively prime condition again for the roach graph.

Suppose we are labeling a cycle, C_n using consecutive odd integers. Let the label of the initial vertex be $f(V_1)=k>1$, and let the label of the final vertex be $f(V_n)$.

E. Theorem 2

Any cycle graph, C_n can be labeled as the $\gcd(f(V_1), f(V_n))=1$ if the number of interior points, $n-2\neq(q+1).P[k]-1$; q-0, 1, 2, ... and $P[k]\neq s.d$ with a constant gap, $d\in Z^+$ where P[y] denotes the prime factors of y.

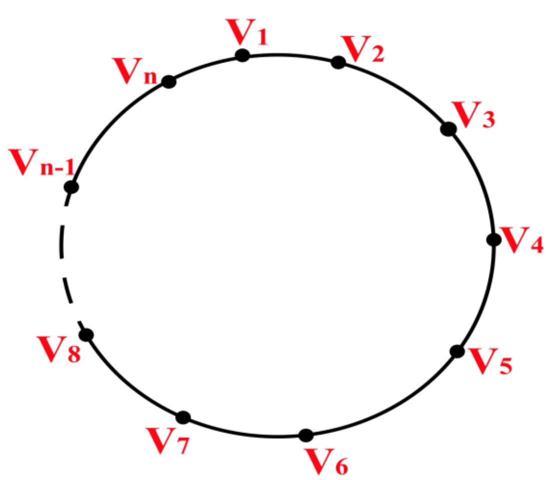

Figure 5: Cycle graph with n vertices

Letthe k-odd prime labeling function of this cycle graph be f.

Proof:

$$f(V_1)=k$$
$$f(V_2)=k+d=f(V_1)+d$$
$$f(V_3)=k+2d=f(V_2)+d$$
$$\vdots$$
$$f(V_{n-1})=k+(n-2)d=f(V_{n-2})+d$$
$$f(V_n)=f(V_{n-1})+d$$

Considering the greatest common divisor of $f(V_1)$andf(V_n),
$\gcd(f(V_1),f(V_n))=\gcd(k,k+(n-1)d)=\gcd(k,(n-1)d)$. If $\gcd(f(V_1),f(V_n))=1$, then prime factors of k must not be an integer multiple of (n-1) and d.

So, $r.P[k]\neq(n-1)$; $r\in Z^+$ and $s.P[k]\neq d;s\in Z^+$

Now, we can write this as $r.P[k]-1\neq n-2$ and $s.P[k]\neq d$.

$n-2\neq r.P[k]-1$; $r\in Z^+$ and $s.P[k]\neq d$

$n-2\neq(q+1).P[k]-1;q\geq0$, $q\in Z$;k>1 and $s.P[k]\neq d$

Thus, if we need to satisfy the condition that $\gcd(f(V_1),f(V_n))=1$, then the number of interior points must not be equal to the value of $(q+1).P[k]-1$ and $s.P[k]\neq d$ for any $d\in Z^+$.

Now, let's apply Theorem 1 regarding ladder graphs by combining Theorem 2 to get some interesting results for the connected, L number of parallel ladder graphs.

F. Theorem 3

Connected L number of parallel ladder graphs, $P_2\times P_{m_i}=L_{m_i}$ are k - odd prime if $2(m_r-1)\neq(q+1).P[k+4\sum_{j=1}^{r-1} m_j +2]-1$;r=1, 2, 3, ...,L-1; q-0, 1, 2, ... where i=1, 2, 3, ...,L.

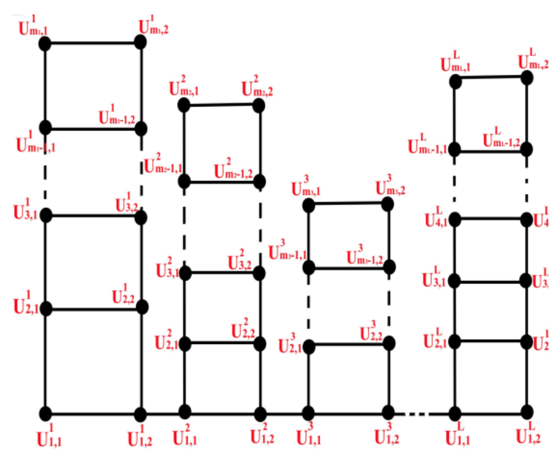

Figure 6: Connected L number of parallel ladder graphs

For the easiness,we used the same labeling pattern used for labeling the grid graphsin Fig. 3 to understand how the ladder graphs have beenconnected to construct this kind of graph.

Follow the labeling steps for each ladder as follows.Start the labeling from the vertex $U^i_{1,1}$, and then continue the labeling for $U^i_{1,2}$, $U^i_{2,1}$, $U^i_{2,1}$,$U^i_{2,2}$, $U^i_{3,1}$, ..., $U^i_{m_i,1}$, $U^i_{m_i,2}$. Repeat this order for each i=1, 2, 3, ..., L. Labeling is done using consecutive odd integers.

But we have to consider a few more things in order to get the labeling properly. After labeling the first ladder, L=1, next we move to the second ladder, L=2, and start labeling from the vertex $U^2_{1,1}$. Here, we must check whether the next consecutive odd integer is relatively prime with the adjacent label of the $U^1_{1,2}$. For this, we can use Theorem 2. And here's how.

To use Theorem 2, basically, we need a cycle graph.

We attach the following graph to figure out this cycle easily. The cycle we consider to determine the label

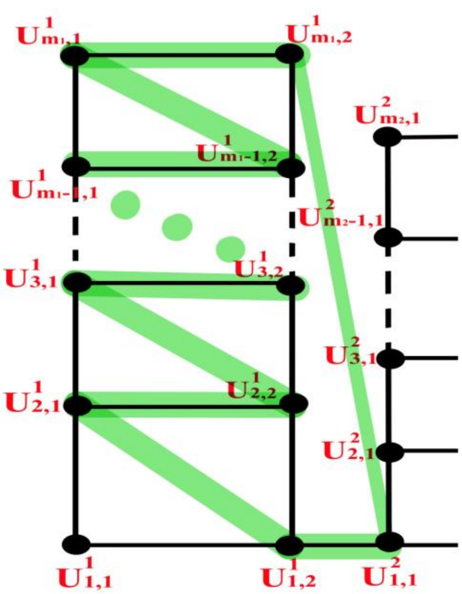

Figure 7: The cycle which we apply Theorem 2

of the vertex $U^2_{1,1}$ is as follows,

It goes through the consecutive odd integers in the following order, $U^1_{1,2}$, $U^1_{2,1}$, $U^2_{1,1}$,$U^1_{2,2}$, $U^1_{3,1}$, $U^1_{3,2}$,…,$U^1_{m_1-1,1}$, $U^1_{m_1,1}$, $U^1_{m_1,2}$,$U^2_{1,1}$.

According to Theorem 2, $V_1 \equiv U^i_{1,2}$, $V_n \equiv U^{i+1}_{1,1}$, and d=2 as we use consecutive odd integers for the labeling for each cycle. Since d=2 and here we use only odd integers for the labeling, s.P[k]≠d.

Suppose we start the labeling from $f(U^1_{1,1})=k>1$, then $f(U^1_{1,2})=k+2$.

To $\gcd(f(U^1_{1,2}), f(U^2_{1,1}))=1$, we need to consider the cycle $U^1_{1,2}$, $U^1_{2,1}$,$U^1_{2,2}$, $U^1_{3,1}$, $U^1_{3,2}$,...,$U^1_{m_1-1,1}$, $U^1_{m_1,1}$, $U^1_{m_1,2}$, $U^2_{1,1}$ and according to Theorem 2, $V_1 \equiv U^1_{1,2}$, $V_n \equiv U^2_{1,1}$.

Number of interior vertices of the above cycle = $2(m_1-1)$ and according to Theorem 2, $2(m_1-1)$ must not be $2(m_1-1) \neq (q+1).P[k+2]-1$; q-0, 1, 2, ...

To $\gcd(f(U^2_{1,2}), f(U^3_{1,1}))=1$, we need to consider the cycle $U^2_{1,2}$, $U^2_{2,1}$,$U^2_{2,2}$, $U^2_{3,1}$, $U^2_{3,2}$,...,$U^2_{m_1-1,1}$, $U^2_{m_1,1}$, $U^2_{m_1,2}$, $U^3_{1,1}$ and according to Theorem 2, $V_1 \equiv U^2_{1,2}$, $V_n \equiv U^3_{1,1}$.

Number of interior vertices of the above cycle = $2(m_2-1)$ and according to Theorem 2, $2(m_2-1)$ must not be $2(m_2-1) \neq (q+1).P[k+4m_1+2]-1$; q-0, 1, 2, ...

To $\gcd(f(U^3_{1,2}), f(U^4_{1,1}))=1$, we need to consider the cycle $U^3_{1,2}$, $U^3_{2,1}$,$U^3_{2,2}$, $U^3_{3,1}$, $U^3_{3,2}$,...,$U^3_{m_1-1,1}$, $U^3_{m_1,1}$, $U^3_{m_1,2}$, $U^4_{1,1}$ and according to Theorem 2, $V_1 \equiv U^3_{1,2}$, $V_n \equiv U^4_{1,1}$.

Number of interior vertices of the above cycle = $2(m_3-1)$ and according to Theorem 2, $2(m_2-1)$ must not be $2(m_3-1) \neq (q+1).P[k+4(m_1+m_2)+2]-1$; q-0, 1, 2, ...

Likewise, we have to fulfill the requirement of the $\gcd(f(U^r_{1,2}), f(U^{r+1}_{1,1}))=1$ to label connected, L number of parallel ladder graphs odd prime where r=1, 2, 3, ...,L-1.

For each i, the cycle needs to consider is $U^r_{1,2}$, $U^r_{2,1}$,$U^r_{2,2}$, $U^r_{3,1}$, $U^i_{3,2}$,...,$U^r_{m_1-1,1}$, $U^r_{m_1,1}$, $U^r_{m_1,2}$, $U^{r+1}_{1,1}$ and corresponding starting and ending vertices are $V_1 \equiv U^r_{1,2}$, $V_n \equiv U^{r+1}_{1,1}$. Therefore, for each r, the condition that needs to satisfy is $2(m_r-1) \neq (q+1).P[k+4\sum_{j=1}^{r-1} m_j +2]-1$;r=1, 2, 3, ...,L-1;

q-0,1, 2, ...

III. RESULTS AND DISCUSSION

In this research paper, we mainly discuss k - odd prime labeling of m×ngrid graphs and disjoint union of grid graphs, an irregular chain of grid graphs and connected L number of parallel ladder graphs, and roach graphs. Further, we prove some theoremsthat are needed in finding the k - odd prime labeling of the above-mentioned graphs as well.

In Theorem 1, we introduce the k- odd prime labeling for m×n grid graph where it is k - odd prime graph whenever $n=2^x$ for any odd k value. Then, we discuss two obvious results that can be obtained by using Theorem 1 under remarks 1, 2, and 3. Remark 3 discusses the k - odd prime labeling of the roach graph which can be easily obtained by using the ladder graph of the same dimensions. k- odd prime labeling of the ladder graphs can be obtained using the m×n grid graphs in Fig. 1 when n=2.

In Theorem 2, we introduce a new theorem for determining the cycle length depending on the label of the initial vertex of a cycle graphwhenever we label a cycle graph using consecutive odd integers.
Finally, we introduce Theorem 3 to figure out what kind of requirements must be satisfied to obtaink - odd prime labeling of connectedL parallel ladder graphsusing the results we obtained in Theorem 2.

CONCLUSION

Graph labeling is one of the most engrossing research areas in graph theory.In this research, we carried out a consecutive k– odd prime labeling for m×n grid graph and disjoint union of grid graphs, an irregular chain of grid graphs, and connected L number of parallel ladder graphs, and roach graphs.

In future work, we hope to obtain more results of k-odd prime labeling of vivid classes of graphs using Theorem 2 such as connected L number of parallel m×n grid graphs.

REFERENCES

[1] M. Zakaria Youssef and Z. Saad Almoreed, “On odd prime labeling of graphs,” *Open J. Discret. Appl. Math*, vol. 3, no. 3, pp. 33–40, Oct. 2020, doi: 10.30538/psrp-odam2020.0041.

[2] H. Carter and N. B. Fox, “Odd Prime Graph Labelings,” *ArXiv*, Aug. 2022, Accessed: Oct. 25, 2022. [Online]. Available: https://doi.org/10.48550/arXiv.2208.08488

[3] G. Gajalakshmi and S. Meena, “On Odd Prime Labelings of Snake Related Graphs,” *J Algebr Stat*, vol. 13, no. 1, pp. 630–634, 2022, [Online]. Available: https://publishoa.com

[4] P. U. M, S. K. P, and U. M. Prajapati, “International Journal of Scientific Research and Reviews Odd Prime Labeling of Various Snake Graphs,” *IJSRR*, vol. 2019, no. 2, pp. 2876–2885, [Online]. Available: www.ijsrr.org

[5] A. H. Berliner, J. Hook, A. Mbirika, N. Dean, A. Marr, and C. D. Mcbee, “Coprime and Prime Labelings of Graphs,” 2016