Convergence of the Newton-Raphson's Method

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Abstract- Using the Newton-Raphson method, we have computed the fourth roots of natural numbers ranging from 1 to 30 and compared these results with the actual values. Our analysis reveals that the smallest error and percentage error (both being zero) were observed when calculating the fourth roots of 1 and 16. The average error was found to be 0.000000094927, while the maximum error and maximum percentage error were 0.000001154116 and 0.000097049224, respectively, for the fourth root of 2. The average percentage error is 0.000006303776. In general, the rate of numerical convergence for determining the fourth roots of numbers from 1 to 30 using the Newton-Raphson method decreases as the number increases.

Indexed Terms- Convergence, Newton-Raphson method, Numerical accuracy, Iteration, Stopping tolerance, Approximation.

I. INTRODUCTION

In prior research, we have presented two papers titled "Convergence of the Bisection Method" and "Convergence of the False Position Method" in the journal "The Scientific Temper" [1,2]. This study now delves into the convergence properties of the Newton-Raphson method, also known as the Newton Method. The Newton-Raphson method is a potent numerical equation-solving technique rooted in the concept of linear approximation, offering remarkable efficiency in finding roots [3,4]. This method, along with various derivatives, follows a path of successive linear approximations that ultimately converge to the solution of the original nonlinear problem [5-9].

To gain a deeper understanding, let's initially consider the one-dimensional case. Starting with an initial guess for the root, x1, the function is approximated through its first-order Taylor expansion at x1. This approximation corresponds to a tangent line to the function's graph at x1. The updated estimate of the root, x2, becomes the root of this tangent line. This process continues iteratively, with x3 becoming the root of the tangent line at x2. This iteration continues until the roots converge [10-17].

In the multivariate case, the method involves obtaining successive root estimates by approximating the function f with a first-order Taylor expansion centered on the current estimate. Given the current estimate, xn, the updated estimate, xn+1, is determined by solving the linear system of equations:

$$f(x) = f(xn) + f'(xn) (x - xn) = 0.$$

Thus,
$$xn+1 = xn - f(xn) / f'(xn) \quad (1).$$

It's important to note that, in the n-dimensional scenario, this problem should not be addressed by inverting the Jacobian matrix and performing matrix multiplication with f(xn).

II. MATERIALS AND METHODS

The methodology applied in this study revolves around the Newton-Raphson method, which involves the assessment of the function's slope, represented as the tangent line, at the present location. This tangent line's zero is then utilized as the subsequent reference point in the iterative process, persisting until the root of the function is successfully determined.

While the Newton-Raphson method is notably more efficient when compared to other "basic" techniques such as the Bisection method, it is worth noting that this method necessitates the computation of the derivative of the function at the reference point. This particular aspect of the procedure can pose difficulties in certain cases. Moreover, the tangent line at times exhibits erratic behavior and may occasionally become ensnared in a looping pattern. As a result, the exceptional efficiency initially promised may not always be fully realized. In light of these potential challenges, it is advisable to closely monitor the steps generated by the Newton-Raphson method. Should the step size become excessively large or the values oscillate, it is recommended to consider alternative, more conservative approaches to address the specific case [18-24].

Computer program developed in C++ programming language for the calculation of roots of an equation is as follows-

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
//Newton-Raphson method to find fourth roots of 1
void main(void)
{
  FILE *fpt;
int n;
floatdelta,ar[1000],aa;
double f(float x);
doublefdash(float x);
clrscr();
  //Filename to store result
fpt=fopen("lavnr1.txt", "w");
  //ar[0] is the initial guess of root
ar[0]=0.1; n=0;
  //Value of function f(x)
fprintf(fpt,"f(x)=x^4-25\n");
  //delta is tolerence
delta=0.00001;
fprintf(fpt,"    n       ar[n]             f(ar[n])\n");
printf("   n       ar[n]             f(ar[n])\n");
            do
            {
            ar[n+1]=ar[n]-f(ar[n])/fdash(ar[n]);
            aa=fabs(f(ar[n+1]));
            n++;
            fprintf(fpt,"%3d                              %15.12f
%15.12f\n",n,ar[n],f(ar[n]));
            printf("%3d                               %15.12f
%15.12f\n",n,ar[n],f(ar[n]));
            } while (aa> delta);
printf("Root= %20.12f\n",ar[n]);
printf("Value of function=%20.12f\n", f(ar[n]));
printf("No. of iterations=%3d\n",n);
getch();
fclose(fpt);
 }
//Function definition
double f(float x)
{
double r;
  r=x*x*x*x-1;
return(r);
}
//Differential of function
doublefdash(float x)
{
double r;
  r=4*x*x*x;
return(r);
}
```

With the help of above computer program, fourth roots of the number from 1 to 30with initial guess 0.1 have been calculated. For this, the following functions have been taken

$f(x) = x^4 - n$ where n = 1,2, 3, ……. , 30

Numerical accuracy of Newton-Raphson method has been measured by percentage error and defined as follows–

Percentage error = error in the value of fourth root * 100/actual value of fourth root

Numerical accuracy of the Newton-Raphson method is inversely proportional to percentage error.

Numerical rate of convergence of Newton-Raphson method = 1 / (1000αβγ)

where   α = Total number of iterations

β = 30 * initial guess of root

γ = Stopping tolerance

III. RESULT AND DISCUSSION

- Calculation of fourth root of 1 by the Newton-Raphson method

The Newton-Raphson method has been used to calculate the root of equation

$f(x) = x^4 - 1 = 0$

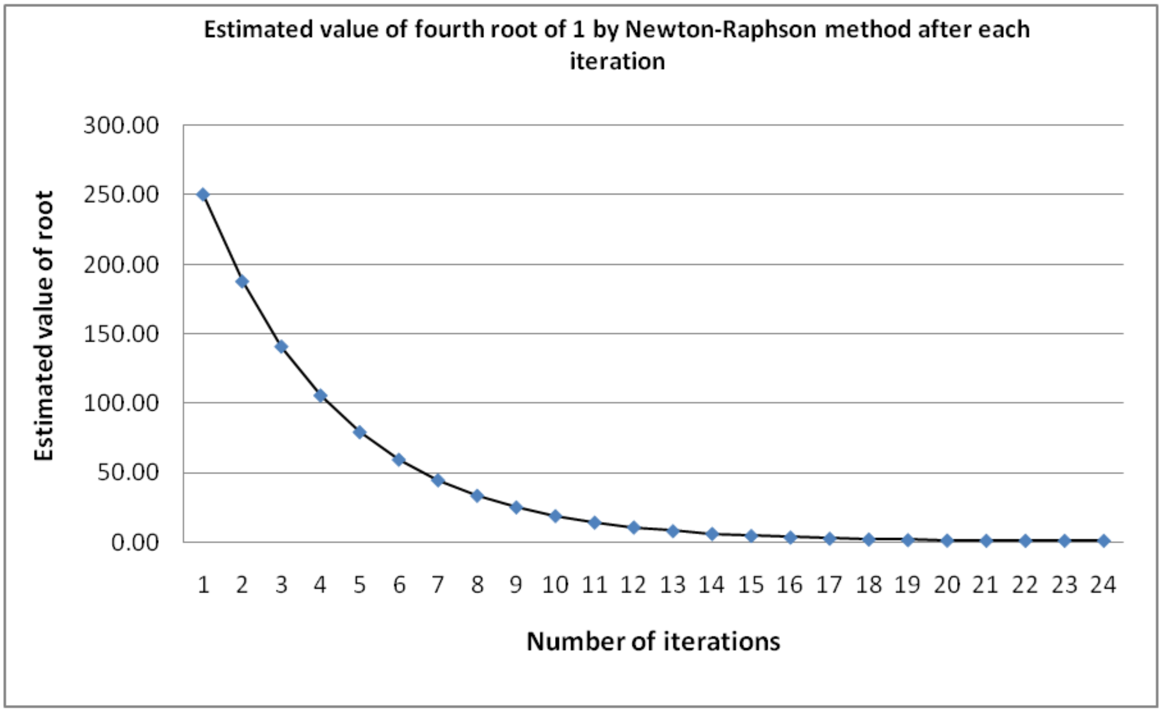
with initial guess of $x_0 = 0.1$ by using C++ computer program. Number of iterations, root guessed by the Newton-Raphson method in each iteration (x_n) and value of function at $x = x_n$is given in Table-1. Root guessed by the Newton-Raphson method after each iterationis shown in Graph-1.

Table-1: Root guessed by the Newton-Raphson method in each iteration (x_n) and value of function $f(x) = x^4 - 1$at at$x=x_n$

No. of iteration	Root guessed by the Newton-Raphson method (x_n)	Value of function at $x=x_n$ *i. e.* $f(x_n)$
1	250.074981689453	3910938463.357570000000
2	187.556243896484	1237445573.834880000000
3	140.667190551758	391535597.855772000000
4	105.500396728516	123884327.493161000000
5	79.125297546387	39197774.812289400000
6	59.343975067139	12402421.847590400000
7	44.507984161377	3924204.113129460000
8	33.380989074707	1241642.165968030000
9	25.035749435425	392863.136867540000
10	18.776828765869	124304.110246735000
11	14.082659721375	39330.339948690000
12	10.562084197998	12444.103161413400
13	7.921775341034	3937.130351975160
14	5.941834449768	1245.471031092210
15	4.457567691803	393.813312559357
16	3.345998287201	124.343800101109
17	2.516172409058	39.083125938481
18	1.902822732925	12.109717254544
19	1.463403582573	3.586236597327
20	1.177324175835	0.921251628497
21	1.036190629005	0.152812405653
22	1.001852273941	0.007429706708
23	1.000005125999	0.000020504155
24	1.000000000000	0.000000000000

Actual value of fourth root of 1	1.000000000000
Calculated value of fourth root of 1 by Newton-Raphson method	1.000000000000
Difference between actual and calculated values of fourth root of 1 by Newton-Raphson method	0.000000000000
Percentage error in the value of fourth root of 1 calculated by Newton-Raphson method	0.000000000000
Numerical rate of convergence of Newton-Raphson method in the calculation of fourth root of 1	1.388888888889

Graph-1: Root guessed by the Newton-Raphson method in the equation $x^4 - 1 = 0$



Calculation of fourth root of 2 by the Newton-Raphson method

The Newton-Raphson method has been used to calculate the root of equation
$f(x) = x^4 - 2 = 0$
with initial guess of $x_0 = 0.1$ by using C++ computer program. Number of iterations, root guessed by the Newton-Raphson method in each iteration (x_n) and value of function at $x = x_n$is given in Table-2. Root guessed by the Newton-Raphson method after each iterationis shown in Graph-2.

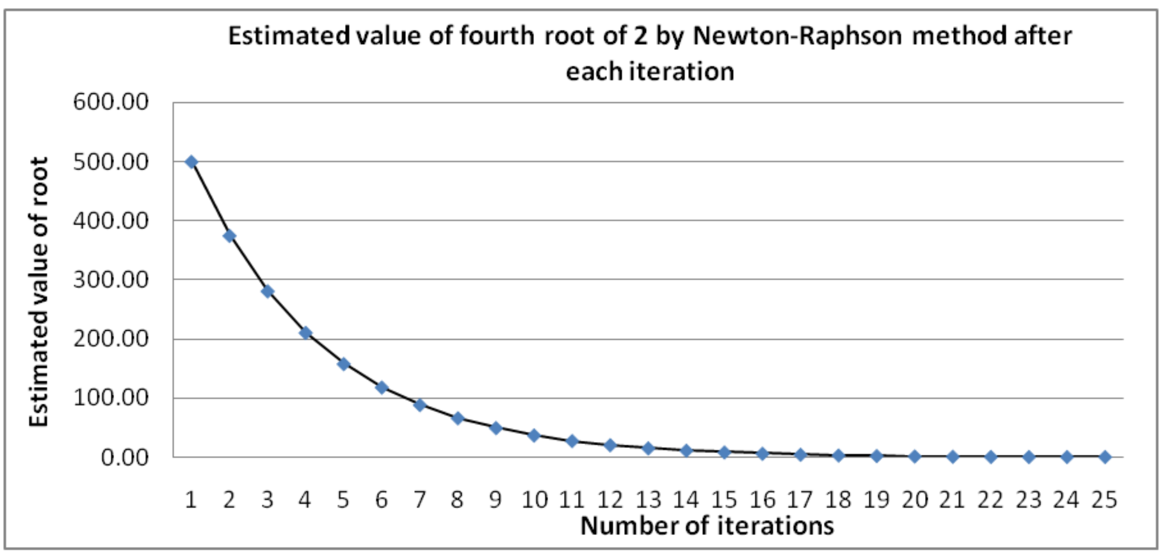
Table-2: Root guessed by the Newton-Raphson method in each iteration (x_n) and value of function $f(x) = x^4 - 2$at at$x=x_n$

No. of iteration	Root guessed by the Newton-Raphson method (x_n)	Value of function at $x=x_n$ i. e. $f(x_n)$
1	500.074981689453	62537499276.950300000000
2	375.056243896484	19787257239.279900000000
3	281.292175292969	6260811180.259070000000
4	210.969131469727	1980959786.136660000000
5	158.226852416992	626788116.410148000000
6	118.670143127441	198319701.590961000000
7	89.002609252930	62749597.093287200000
8	66.751960754395	19854367.876608000000
9	50.063972473145	6282045.676109990000
10	37.547985076904	1987678.359152300000
11	28.160997390747	628913.255375148000
12	21.120769500732	198991.526190695000

No. of iteration	Root guessed by the Newton-Raphson method (x_n)	Value of function at $x=x_n$ i. e. $f(x_n)$
13	15.840630531311	62961.644504997500
14	11.880599021912	19920.936631488300
15	8.910747528076	6302.585737668240
16	6.683767318726	1993.654196839160
17	5.014500141144	630.281669704365
18	3.764840602875	198.902978040374
19	2.833000183105	62.414910893650
20	2.146740436554	19.238221574373
21	1.660594820976	5.604220760033
22	1.354635119438	1.367358247330
23	1.217118501663	0.194479140750
24	1.190152645111	0.006368333123
25	1.189208269119	0.000007763951

Actual value of fourth root of 2	1.189207115003
Calculated value of fourth root of 2 by Newton-Raphson method	1.189208269119
Difference between actual and calculated values of fourth root of 2 by Newton-Raphson method	0.000001154116
Percentage error in the value of fourth root of 2 calculated by Newton-Raphson method	0.000097049224
Numerical rate of convergence of Newton-Raphson method in the calculation of fourth root of 2	1.333333333333

Graph-2: Root guessed by the Newton-Raphson method in the equation $x^4 - 2 = 0$



Consolidated analysis of the fourth roots of numbers from 1 to 30 calculated by Newton-Raphson method

The value of fourth root, error in the determination of fourth root, percentage error and numerical rate of convergence in the Newton-Raphson method are shown in Table-3(a) and Table-3(b). The actual value of fourth root and the value of fourth root calculated by Newton-Raphson method are shown in Graph-3. Error in the value of fourth root calculated by Newton-Raphson methodis given in Graph-4. Percentage error in the values of fourth root calculated by Newton-Raphson methodis given in Graph-5. Numerical rate of convergence in the determination ofthe fourth roots by Newton-Raphson methodis given in Graph-6.

Table-3(a): Actual value of fourth root, value of fourth root calculated by Newton-Raphson method and error in the determination of fourth root by Newton-Raphson method in findingthe roots of equations
$f(x) = x^4 - n = 0; n = 1, 2, \ldots, 30$

S. No.	Function	No. of Iterations	Actual value of fourth root	Value of fourth root calculated by Newton-Raphson method	Error in the fourth root calculated by Newton-Raphson method
1	$f(x)=x^4-1$	24	1.000000000000	1.000000000000	0.000000000000
2	$f(x)=x^4-2$	25	1.189207115003	1.189208269119	0.000001154116
3	$f(x)=x^4-3$	27	1.316074012952	1.316074013710	0.000000000758
4	$f(x)=x^4-4$	27	1.414213562373	1.414213776588	0.000000214215
5	$f(x)=x^4-5$	28	1.495348781221	1.495348811150	0.000000029929
6	$f(x)=x^4-6$	28	1.565084580073	1.565084934235	0.000000354162
7	$f(x)=x^4-7$	29	1.626576561698	1.626576542854	0.000000018844
8	$f(x)=x^4-8$	29	1.681792830507	1.681792855263	0.000000024756
9	$f(x)=x^4-9$	30	1.732050807569	1.732050776482	0.000000031087
10	$f(x)=x^4-10$	30	1.778279410039	1.778279423714	0.000000013675
11	$f(x)=x^4-11$	30	1.821160286838	1.821160316467	0.000000029629
12	$f(x)=x^4-12$	30	1.861209718204	1.861209750175	0.000000031971
13	$f(x)=x^4-13$	31	1.898828922116	1.898828864098	0.000000058018
14	$f(x)=x^4-14$	31	1.934336420268	1.934336423874	0.000000003606
15	$f(x)=x^4-15$	31	1.967989671265	1.967989683151	0.000000011886
16	$f(x)=x^4-16$	31	2.000000000000	2.000000000000	0.000000000000
17	$f(x)=x^4-17$	31	2.030543184869	2.030543088913	0.000000095956
18	$f(x)=x^4-18$	31	2.059767143907	2.059767246246	0.000000102339
19	$f(x)=x^4-19$	32	2.087797629930	2.087797641754	0.000000011824
20	$f(x)=x^4-20$	32	2.114742526881	2.114742517471	0.000000009410
21	$f(x)=x^4-21$	32	2.140695142928	2.140695095062	0.000000047866
22	$f(x)=x^4-22$	32	2.165736770668	2.165736675262	0.000000095406
23	$f(x)=x^4-23$	32	2.189938703095	2.189938783646	0.000000080551
24	$f(x)=x^4-24$	32	2.213363839401	2.213363885880	0.000000046479
25	$f(x)=x^4-25$	32	2.236067977500	2.236068010330	0.000000032830
26	$f(x)=x^4-26$	32	2.258100864353	2.258100986481	0.000000122128
27	$f(x)=x^4-27$	33	2.279507056955	2.279507160187	0.000000103232
28	$f(x)=x^4-28$	33	2.300326633791	2.300326585770	0.000000048021
29	$f(x)=x^4-29$	33	2.320595787106	2.320595741272	0.000000045834
30	$f(x)=x^4-30$	33	2.340347319321	2.340347290039	0.000000029282
Average value					0.000000094927

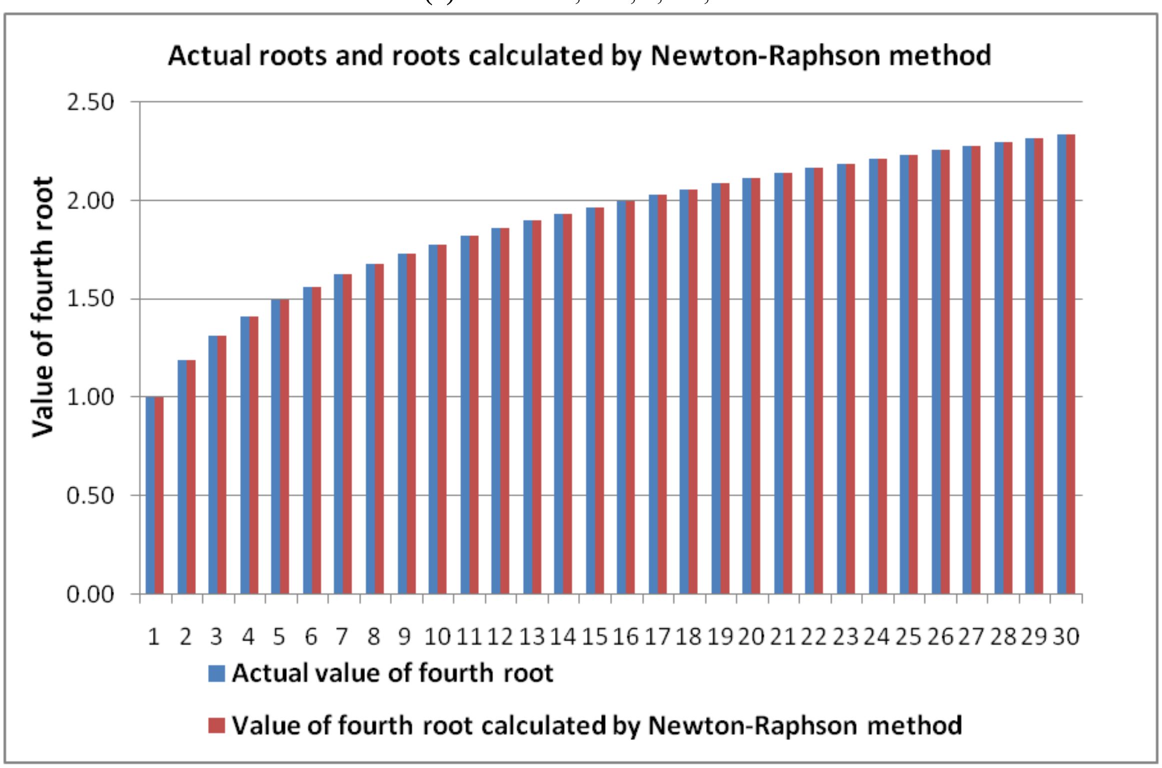
S. No.	Function	No. of Iterations	Actual value of fourth root	Value of fourth root calculated by Newton-Raphson method	Error in the fourth root calculated by Newton-Raphson method
Minimum value					0.000000000000
Maximum value					0.000001154116

Table-3(b): Actual value of fourth root, percentage error in the calculation of fourth root and numerical rate of convergence of Newton-Raphson method in the determination of roots of equations $f(x) = x^4 - n = 0; n = 1, 2, \ldots, 30$

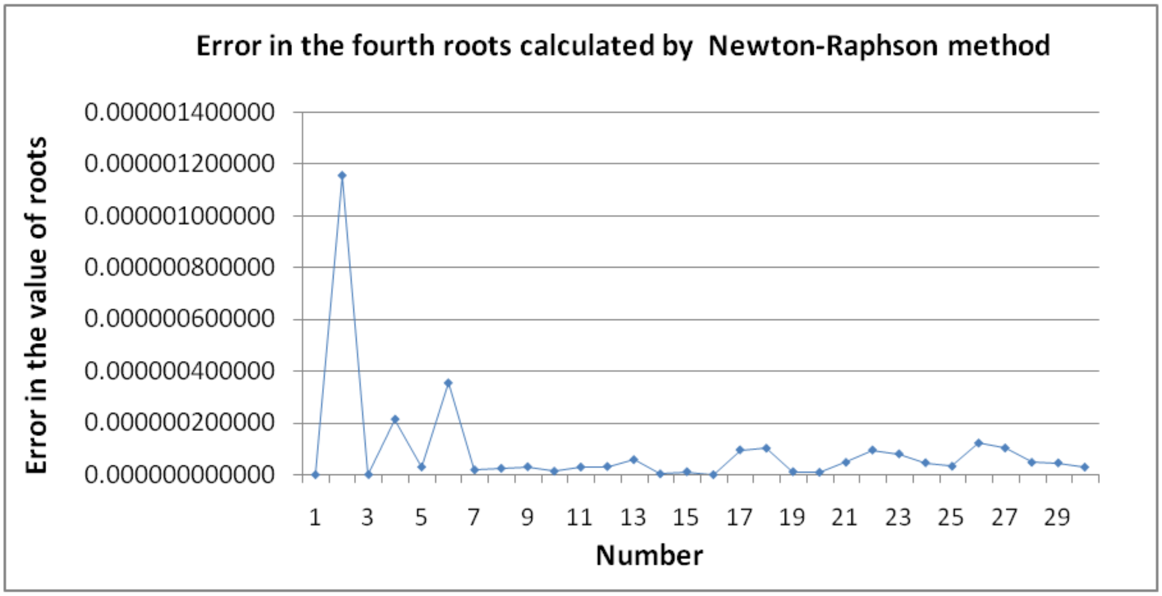
S. No.	Function	No. of Iterations	Actual value of fourth root	Percentage error in the fourth root calculated by Newton-Raphson method	Numerical rate of convergence of by Newton-Raphson method
1	$f(x)=x^4-1$	24	1.000000000000	0.000000000000	1.388888888889
2	$f(x)=x^4-2$	25	1.189207115003	0.000097049224	1.333333333333
3	$f(x)=x^4-3$	27	1.316074012952	0.000000057558	1.234567901235
4	$f(x)=x^4-4$	27	1.414213562373	0.000015147281	1.234567901235
5	$f(x)=x^4-5$	28	1.495348781221	0.000002001458	1.190476190476
6	$f(x)=x^4-6$	28	1.565084580073	0.000022628918	1.190476190476
7	$f(x)=x^4-7$	29	1.626576561698	0.000001158494	1.149425287356
8	$f(x)=x^4-8$	29	1.681792830507	0.000001471975	1.149425287356
9	$f(x)=x^4-9$	30	1.732050807569	0.000001794802	1.111111111111
10	$f(x)=x^4-10$	30	1.778279410039	0.000000769006	1.111111111111
11	$f(x)=x^4-11$	30	1.821160286838	0.000001626937	1.111111111111
12	$f(x)=x^4-12$	30	1.861209718204	0.000001717743	1.111111111111
13	$f(x)=x^4-13$	31	1.898828922116	0.000003055459	1.075268817204
14	$f(x)=x^4-14$	31	1.934336420268	0.000000186438	1.075268817204
15	$f(x)=x^4-15$	31	1.967989671265	0.000000603945	1.075268817204
16	$f(x)=x^4-16$	31	2.000000000000	0.000000000000	1.075268817204
17	$f(x)=x^4-17$	31	2.030543184869	0.000004725629	1.075268817204
18	$f(x)=x^4-18$	31	2.059767143907	0.000004968469	1.075268817204
19	$f(x)=x^4-19$	32	2.087797629930	0.000000566346	1.041666666667
20	$f(x)=x^4-20$	32	2.114742526881	0.000000444977	1.041666666667
21	$f(x)=x^4-21$	32	2.140695142928	0.000002236006	1.041666666667
22	$f(x)=x^4-22$	32	2.165736770668	0.000004405244	1.041666666667
23	$f(x)=x^4-23$	32	2.189938703095	0.000003678238	1.041666666667
24	$f(x)=x^4-24$	32	2.213363839401	0.000002099942	1.041666666667

S. No.	Function	No. of Iterations	Actual value of fourth root	Percentage error in the fourth root calculated by Newton-Raphson method	Numerical rate of convergence of by Newton-Raphson method
25	$f(x)=x^4-25$	32	2.236067977500	0.000001468212	1.041666666667
26	$f(x)=x^4-26$	32	2.258100864353	0.000005408429	1.041666666667
27	$f(x)=x^4-27$	33	2.279507056955	0.000004528708	1.010101010101
28	$f(x)=x^4-28$	33	2.300326633791	0.000002087582	1.010101010101
29	$f(x)=x^4-29$	33	2.320595787106	0.000001975100	1.010101010101
30	$f(x)=x^4-30$	33	2.340347319321	0.000001251170	1.010101010101
Average values				0.000006303776	1.104698523392
Minimum values				0.000000000000	1.010101010101
Maximum values				0.000097049224	1.388888888889

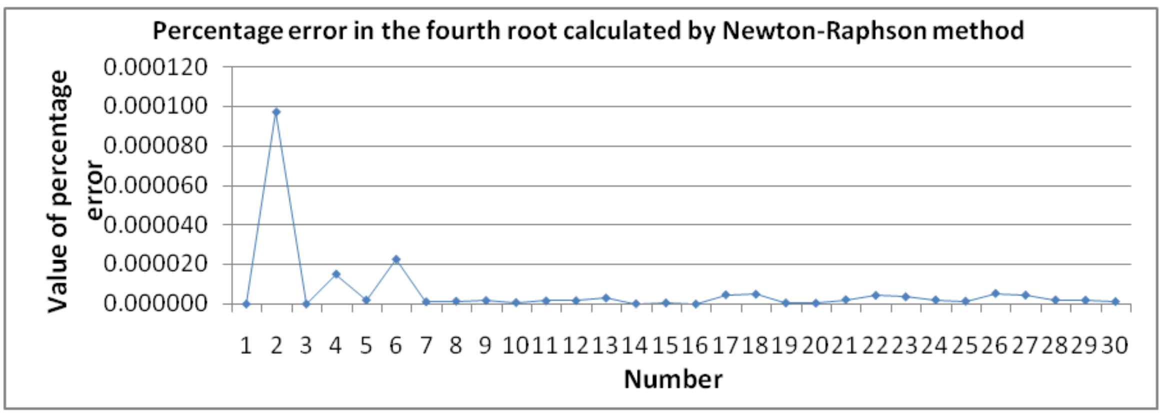
Graph-3: Actual value of fourth root and the value of root calculated by Newton-Raphson method in the equations $f(x) = x^4 - n=0; n=1, 2, …., 30$



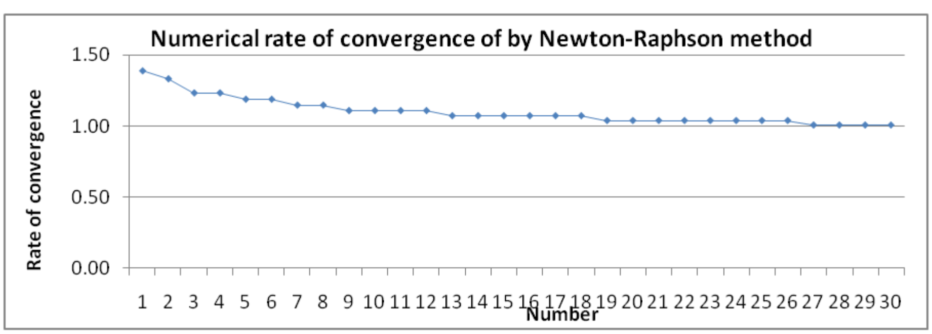
Graph-4: Error in the value of fourth root calculated by Newton-Raphson method in the equations $f(x) = x^4 - n=0$; n=1, 2, …., 30



Graph-5: Percentage error in the value of fourth root calculated by Newton-Raphson method in the equations $f(x) = x^4 - n=0$; n=1, 2, …., 30



Graph-6: Numerical rate of convergence in the determination of the fourth root by Newton-Raphson method in the equations $f(x) = x^4 - n=0$; n=1, 2, …., 30



CONCLUSION

Fourth roots of the natural numbers from 1 to 30 have been found by Newton-Raphson method and these values have been compared with the actual values. The minimum error zero and minimum percentage error zero has been obtained in the determination of fourth roots of 1 and 16. The average value in the error is 0.000000094927. The maximum error 0.000001154116 and maximum percentage error 0.000097049224have been obtained in the determination of fourth roots of 2. The average value of percentage error is 0.000006303776. The normal trend in the fourth roots as obtained by the method of false position is that the error and percentage error in the roots increase as the number increase.

Generally, numerical rate of convergence in the determination of the fourth root of numbers from 1 to 30 by Newton-Raphson method decreases as the number increases. Minimum, Maximum and average values of the numerical rate of convergence are 1.010101010101, 1.388888888889 and 1.104698523392 respectively.

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LITERATURE REVIEW

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