Coefficient Estimates for a Combined Subclass of Classes of Convex Functions and Starlike Functions

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***Abstract-* We introduce a class of analytic functions and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this class with special character that it tends to the class of Starlike functions as $\alpha \to \frac{\pi}{2}$.**

***Indexed Terms-* Univalent functions, Starlike functions, Close to convex functions and bounded functions.**

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\boldsymbol{\mathcal{S}}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([1], [2]) proved that $|a_2| \le 2$ for the functions $f(z) \in \boldsymbol{\mathcal{S}}$. In 1923, Löwner proved that $|a_3| \le 3$ for the functions $f(z) \in \boldsymbol{\mathcal{S}}$..

With the known estimates $|a_2| \le 2$and $|a_3| \le 3$, it was expected to try to find some relation between a_3 and ${a_2}^2$ for the class $\boldsymbol{\mathcal{S}}$,Fekete and Szegö [5] used Löwner's method to prove the following well known result for the class $\boldsymbol{\mathcal{S}}$.

Let $f(z) \in \boldsymbol{\mathcal{S}}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu, if \ \mu \le 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if \ 0 \le \mu \le 1; \\ 4\mu - 3, if \mu \ge 1. \end{cases} \qquad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ ([7], [8], [18]-[43]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$

and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$

and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \qquad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \mathbb{E}\right\} \qquad (1.6)$$

$$\mathcal{K}(A,B) = \left\{ f(z) \in \mathcal{A}; \frac{\left(zf'(z)\right)'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \mathbb{E} \right. \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\ (A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as

$$\left\{ f(z) \in \mathcal{A}; tan\alpha \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)^{1-\beta} \prec (1+tan\alpha)\frac{1+Aw(z)}{1-Bw(z)}; z \in \mathbb{E} \right.$$

and we will denote this class as $\mathcal{K}S^*(\alpha,\beta)$.

We will deal with two subclasses of $KS^*(\alpha,\beta)$ defined as follows in our next paper:

$$\mathcal{K}S^*(\alpha,\beta,A,B) = \left\{ f(z) \in \mathcal{A}; tan\alpha \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \right. \frac{zf'z'}{f'z}^{1-\beta} \prec (1+tan\alpha)\ \frac{1+Az}{1+Bz}; z \in \mathbb{E} \quad (1.8)$$

$$\mathcal{K}S^*(A,B,\alpha,\beta,\gamma) = \left\{ f(z) \in \mathcal{A}; tan\alpha \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \right. \frac{zf'z'}{f'z}^{1-\beta} \prec (1+tan\alpha)\left(\frac{1+Az}{1+Bz}\right)^{\gamma}; z \in \mathbb{E} \quad (1.9)$$

Symbol $\prec$ stands for subordination, which we define as follows:

- Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analyticin $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1. \quad (1.10)$$

It is known that $|d_1| \le 1, |d_2| \le 1 - |d_1|^2$. (1.11)

II. PRELIMINARY LEMMAS

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + - - -. \quad (2.1)$$

III. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in \mathcal{K}S^*(\alpha,\beta)$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{(1+tan\alpha)(A-B)}{\{\beta tan\alpha + 2(1-\beta)\}^2}\left[\dfrac{\beta + [(\beta^2-3\beta)tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+tan\alpha)(A-B)}{\{2\beta tan\alpha + 6(1-\beta)\}} - (A-B)(1+tan\alpha)\mu\right], \\ if \mu \le \dfrac{\beta + [(\beta^2-3\beta)tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+tan\alpha)(A-B) - 4(1-\beta^2-2\beta) - 4\beta tan\alpha + 4\beta^2 tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha + 6(1-\beta)\}}; \quad (3.1) \\ \dfrac{(1+tan\alpha)(A-B)}{6(1-\beta) + 2\beta tan\alpha} \\ if \dfrac{\beta + [(\beta^2-3\beta)tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+tan\alpha)(A-B) - \beta^2 tan^2\alpha - 4 - 4\beta^2 + 8\beta - 4\beta tan\alpha + 4\beta^2 tan\alpha}{(A-B)\qquad(1+tan\alpha)\{2\beta tan\alpha + 6(1-\beta)\}} \\ \le \mu \le \dfrac{\beta + [(\beta^2-3\beta)tan\alpha - 2A - 4 - 4\beta^2 + 8\beta](1+tan\alpha)(A-B) - \beta^2 tan^2\alpha - 4 - 4\beta^2 + 8\beta - 4\beta tan\alpha + 4\beta^2 tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha + 6(1-\beta)\}}; \quad (3.2) \\ \mu(A-B)(1+tan\alpha) - \dfrac{(1+tan\alpha)(A-B)}{\{\beta tan\alpha + 2(1-\beta)\}^2}\left\{\dfrac{\beta + [(\beta^2-3\beta)tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+tan\alpha)(A-B) - \beta^2 tan^2\alpha - 4 - 4\beta^2 + 8\beta - 4\beta tan\alpha + 4\beta^2 tan\alpha}{\{2\beta tan\alpha + 6(1-\beta)\}}\right\} \\ if\ \mu \ge \dfrac{\beta + [(\beta^2-3\beta)tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+tan\alpha)(A-B) - \beta^2 tan^2\alpha - 4 - 4\beta^2 + 8\beta - 4\beta tan\alpha + 4\beta^2 tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha + 6(1-\beta)\}} \quad (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $\mathcal{KS}^*(\alpha,\beta)$, we have

$$tan\alpha\left(\frac{zf'(z)}{f(z)}\right)^{\beta}+\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)^{1-\beta}=(1+tan\alpha)\frac{1+w(z)}{1-w(z)};w(z)\in\mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$tan\alpha\left\{1+\beta a_2z+(2\beta a_3+\frac{\beta(\beta-3)}{2}a_2^2)z^2+--\right.$$
−+1+21−βa2z+2(1−β)(3a3−(β+2)a22)z2+−−−
=(1+tanα)1+A−Bc1z+(A−B)c2−Bc12z2+−−−
. (3.5)

Identifying terms in (3.5), we get

$$a_2=\frac{(A-B)(1+tan\alpha)}{\beta tan\alpha+2(1-\beta)}c_1 \quad (3.6)$$

$$a_3=\frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha+3(2-2\beta)}c_2+\left[\frac{\beta(A-B)(1+tan\alpha)+\beta(\beta-3)tan\alpha-2(A+2)+2\beta(\beta+2)}{2\beta tan\alpha+6-6\beta}\left(\frac{(A-B)^2(1+tan\alpha)^2}{(\beta tan\alpha+2(1-\beta))^2}\right)\right]{c_1}^2. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3-\mu a_2^2=\frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha+6-6\beta}c_2+\frac{\left(\beta(A-B)+\beta(\beta-3)tan\alpha-2(A+2)+2\beta(\beta+2)\right)(A-B)(1+tan\alpha)}{2\beta tan\alpha+6-6\beta}{c_1}^2+\frac{(A-B)(1+tan\alpha)}{(\beta tan\alpha+2(1-\beta))^2}-\frac{\mu(A-B)^2(1+tan\alpha)^2{c_1}2}{(\beta tan\alpha+2(1-\beta))^2}. \quad (3.8)$$

Taking absolute value and using Triangular inequality, (3.8) can be rewritten as

$$|a_3-\mu a_2^2|\le 1-|{c_1}^2|\frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha+6-6\beta}+\frac{\left(\beta(A-B)+\beta(\beta-3)tan\alpha-2(A+2)+2\beta(\beta+2)\right)(A-B)(1+tan\alpha)}{2\beta tan\alpha+6-6\beta}{c_1}^2\frac{(A-B)(1+tan\alpha)}{(\beta tan\alpha+2(1-\beta))^2}-\frac{\mu(A-B)^2(1+tan\alpha)^2{c_1}2}{(\beta tan\alpha+2(1-\beta))^2} \quad (3.9)$$

Using (1.9) in (3.9), Simple calculations yield

$$|a_3-\mu a_2^2|\le \frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha+6-6\beta}+\frac{\left(\beta(A-B)+\beta(\beta-3)tan\alpha-2(A+2)+2\beta(\beta+2)\right)(A-B)(1+tan\alpha)}{2\beta tan\alpha+6-6\beta}{c_1}^2\frac{(A-B)(1+tan\alpha)}{(\beta tan\alpha+2(1-\beta))^2}-\frac{\mu(A-B)^2(1+tan\alpha)^2{c_1}2}{(\beta tan\alpha+2(1-\beta))^2}-{c_1}^2\frac{(1+tan\,\alpha)(A-B)(\beta tan\alpha+2(1-\beta)^2)^2}{(\beta tan\alpha+2(1-\beta))^2(2\beta tan\alpha+6-6\beta)}. \quad (3.10)$$

In this case, (3.10) can be rewritten as

Case I:

$$\mu\le\frac{\beta+[(\beta^2-3\beta)tan\alpha-2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)}{(1+tan\alpha)(A-B)(2\beta tan\alpha+6-6\beta)}.$$

$$|a_3-\mu a_2^2\le\frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha+3(2-2\beta)}+\frac{(1+tan\alpha)(A-B)}{\{\beta tan\alpha+2(1-\beta)\}^2}\left[\frac{\beta+[(\beta^2-3\beta)tan\alpha-2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)}{2\beta tan\alpha+6(1-\beta)}-\mu(1+tan\alpha)\right]|c_1|^2.$$
(3.11)

Subcase I (a):

$$\mu\le\frac{\beta+[(\beta^2-3\beta)tan\alpha-2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)-4(1-\beta^2-2\beta)-4\beta tan\alpha+4\beta^2tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha+6(1-\beta)\}}. \quad (3.11)$$

Using (1.9), (3.11) becomes

$$|a_3-\mu a_2^2|\le\frac{(1+tan\alpha)(A-B)}{\{\beta tan\alpha+2(1-\beta)\}^2}\left[\frac{\beta+[(\beta^2-3\beta)tan\alpha-2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)}{\{2\beta tan\alpha+6(1-\beta)\}}-(A-B)(1+tan\alpha)\mu\right] \quad (3.12)$$

Subcase I (b):

$$\mu \geq \frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)-\beta^2 tan^2\alpha-4-4\beta^2+8\beta-4\beta tan\alpha +4\beta^2 tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha +6(1-\beta)\}}.$$

We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha +6(1-\beta)}. \qquad (3.13)$$

Case II:

$$\mu \geq \frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)}{4(1+tan\alpha)(\beta tan\alpha +3(1-\beta)}$$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{(1+tan\alpha)(A-B)}{6(1-\beta)+2\beta tan\alpha} + \frac{(1+tan\alpha)(A-B)}{\{tan\alpha\beta +2(1-\beta)\}^2}\left[\frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)-\beta^2 tan^2\alpha-4-4\beta^2+8\beta-4\beta tan\alpha +4\beta^2 tan\,\alpha}{\{2\beta tan\alpha +6(1-\beta)\}} - \mu(A-B)(1+tan\alpha)\right]|c_1|^2. \qquad (3.14)$$

Subcase II (a):

$$\mu \leq \frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-4\beta^2+8\beta](1+tan\alpha)(A-B)-\beta^2 tan^2\alpha-4-4\beta^2+8\beta-4\beta tan\alpha +4\beta^2 tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha +6(1-\beta)\}}$$

(3.14) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha +6(1-\beta)} \qquad (3.15)$$

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{(1+tan\alpha)(A-B)}{2\beta tan\alpha +6(1-\beta)}\, if\, \frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)-\beta^2 tan^2\alpha-4-4\beta^2+8\beta-4\beta tan\alpha +4\beta^2 tan\alpha}{(A-B)\qquad (1+tan\alpha)\{\beta tan\alpha +3(1-\beta)\}} \leq \mu \leq \frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-4\beta^2+8\beta](1+tan\alpha)(A-B)-\beta^2 tan^2\alpha-4-4\beta^2+8\beta-4\beta tan\alpha +4\beta^2 tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha +6(1-\beta)\}} \qquad (3.16)$$

Subcase II (b): $\mu \geq \frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)-\beta^2 tan^2\alpha-4-4\beta^2+8\beta-4\beta tan\alpha +4\beta^2 tan\alpha}{(A-B)(1+tan\alpha)\{2\beta tan\alpha +6(1-\beta)\}}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \mu(A-B)(1+tan\alpha) - \frac{(1+tan\alpha)(A-B)}{\{\beta tan\alpha +2(1-\beta)\}^2}\left\{\frac{\beta+[(\beta^2-3\beta)tan\alpha -2A-4-2\beta^2+4\beta](1+tan\alpha)(A-B)-\beta^2 tan^2\alpha-4-4\beta^2+8\beta-4\beta tan\alpha +4\beta^2 tan\alpha}{\{2\beta tan\alpha +6(1-\beta)\}}\right\}. \qquad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$f_1(z) = (1 + az)^b$

Where $a = \frac{2\gamma\{\beta+3(1-\beta)\tan\alpha\}}{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}\}^3-2\gamma}$

And $b = \frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}\}^3-2\gamma}{\{\beta+3(1-\beta)\tan\alpha\}\{\beta+2(1-\beta)\tan\alpha\}\}}$

Extremal function for (3.2) is defined by $f_2(z) = z(1 + cz^2)^d$,
Where $c = \frac{\tan\alpha}{\beta+3(1-\beta)\tan\alpha}$ and $d = \frac{\gamma}{\tan\alpha}$

Corollary 3.2: Putting $\gamma = 1, \beta = 0$ and applying limit as $\alpha \to \frac{\pi}{2}$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1-\mu, if \mu \leq 1; \\ \frac{1}{3} if 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, if\ \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [9] and are results for the class of univalent convex functions.
Corollary 3.3: Putting $\alpha = 0, \beta = 1, \gamma = 0$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu, if \mu \leq \frac{1}{2}; \\ 1 if \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, if\ \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [9] and are results for the class of univalent starlike functions.

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