Bounds of Fekete-Szegö Inequality for a Subclasses of Regular Functions

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***Abstract-* We introduce some classes of analytic functions, its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.**

***Indexed Terms-* Univalent functions, Starlike functions, Close to convex functions and bounded functions, Closed functions, FeketeSzego inequality, Subordination method, Extremal function.**

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra [1], Babalola [6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in$
$\mathcal{A}$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \qquad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \qquad (1.6)$$

$$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\,(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as

$$\left\{f(z) \in \mathcal{A}; \frac{z[((f'(z))^2 + f(z)f''(z)]}{f(z)f'(z)} = \left(\frac{1+w(z)}{1-w(z)}\right); z \in \mathbb{E}\right\}$$

and we will denote this class as $S^*(f^2, f')$.

We will deal with two subclasses of $S^*(f^2, f')$ defined as follows in our next paper:

$$S^*(f^2, f', A, B) = \left\{f(z) \in \mathcal{A}; \frac{z[((f'(z))^2 + f(z)f''(z)]}{f(z)f'(z)} \prec \frac{1+Az}{1+Bz}; z \in \mathbb{E}\right\} \quad (1.8)$$

$$S^*(f^2, f', \delta) = \left\{f(z) \in \mathcal{A}; \frac{z[((f'(z))^2 + f(z)f''(z)]}{f(z)f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}; z \in \mathbb{E}\right\} \quad (1.9)$$

Symbol $\prec$ stands for subordination, which we define as follows:
Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1. \quad (1.10)$$

Known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$. (1.11)

II. PRELIMINARY LEMMAS

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots. \quad (2.1)$$

III. MAIN RESULTS

THEOREM 3.1:Let $f(z) \in S^*(f^2, f')$., Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{9}\left[\frac{19}{4} - 4\mu\right], if \mu \leq \frac{5}{8}; & (3.1) \\ \frac{1}{4} \quad if \frac{5}{8} \leq \mu \leq \frac{7}{4}; & (3.2) \\ \frac{1}{9}\left[-\frac{19}{4} + 4\mu\right], if\ \mu \geq \frac{7}{4} & (3.3) \end{cases}$$

The results are sharp.

PROOF: By definition of $S^*(f^2, f')$, we have

$$\frac{z[((f'(z))^2 + f(z)f''(z)]}{f(z)f'(z)} = \frac{1+w(z)}{1-w(z)}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$z+6a_2z^2 + (12a_3 + 6a_2^2)z^3 + (20a_4 + 20a_2a_3)z^4 + \cdots = z+(3a_2+2c_1)z^2 + [4a_3 + 2a_2^2 + 6a_2c_1 + 2(c_2 + {c_1}^2) +]z^3 + \cdots \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{2}{3}c_1 \quad (3.6)$$

$$a_3 = \frac{1}{4}c_2 + \frac{19}{36}{c_1}^2. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{1}{4}c_2 + \left[\frac{19}{36} - \frac{4}{9}\mu\right]c_1^2. \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}|c_2| + \frac{1}{9}\left|\frac{19}{4} - 4\mu\right| |c_1^2|. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}(1 - |c_1|^2) + \frac{1}{9}\left|\frac{19}{4} - 4\mu\right| |c_1^2$$

$$= \frac{1}{4} + \frac{1}{9}\left(\left|\frac{19}{4} - 4\mu\right| - \frac{9}{4}\right)|c_1|^2. \quad (3.10)$$

CASE I: $\mu \leq \frac{19}{16}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \frac{1}{9}[\frac{5}{2} - 4\mu]|c_1|^2. \quad (3.11)$$

SUBCASE I (A): $\mu \leq \frac{5}{8}$. using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{1}{9}\left[\frac{19}{4} - 4\mu\right]. \quad (3.12)$$

SUBCASE I (B): $\mu \geq \frac{5}{8}$. We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}. \quad (3.13)$$

CASE II: $\mu \geq \frac{19}{16}$

Preceding as in case i, we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \frac{1}{9}[4\mu - 7]|c_1|^2 \quad (3.14)$$

SUBCASE II (A): $\mu \leq \frac{7}{4}$

(3.14) takes the form $|a_3 - \mu a_2^2| \leq \frac{1}{4}$ (3.15)

Combining subcase i (b) and subcase ii (a), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \, if \, \frac{5}{8} \leq \mu \leq \frac{7}{4} \quad (3.16)$$

SUBCASE II (B): $\mu \geq \frac{7}{4}$

Preceding as in subcase i (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{9}\left[4\mu - \frac{19}{4}\right]. \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

CONCLUSION

A subclass of analytic functions which take a broad view of some well-known subclasses of analytic and univalent functions was demarcated. The better estimates for the Fekete-Szegö functional for the defined class were obtained along with extremal functions. The study combines existing results and attains new outcomes in geometric function theory. Forthcoming researches can be done to acquire the geometric properties.

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