Coefficient Bounds for a Subclass of Regular P-Valent Functions

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Abstract- Here we describe some classes of analytic functions and its subclasses by which we will be obtaining sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Indexed Terms- Univalent functions, Starlike functions, Close to convex functions and bounded functions.

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$. In 1916, Bieber Bach ([2]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$,Fekete and Szegö[5] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$
(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (Chhichra, Babalola).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in$
$\mathcal{A}$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \qquad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.5)$$

The class of close to convex functions is denoted by Cand was introduced by Kaplan and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \qquad (1.6)$$

$$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \qquad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\,(A,B)$ is a subclass of $\mathcal{K}$. Several researchers obtained new classes, subclasses and new results ([6]-[66])

We introduce a new subclass as$\left\{f(z) \in \mathcal{A}; \frac{z(zf'(z))'}{z(f'(z))} \prec p\frac{1+w(z)}{1-w(z)}; z \in \mathbb{E}\right\}$ and we will denote this class as $f(z) \in KS_p^*]$,

Symbol $\prec$ stands for subordination, which we define as follows:
Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.8)
It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$. (1.9)

II. PRELIMINARY LEMMAS

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + --- \quad (2.1)$$

III. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in KS_p^*$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \left[\frac{p^2(2p+1)}{p+2} - \frac{\mu 4p^4}{(p+1)^2}\right]; \ if \ \mu \leq \frac{(p+1)^2}{2p(p+2)} & (3.1) \\ \frac{p^2}{(p+2)}; \ if \ \frac{(p+1)^2}{2p(p+2)} \leq \mu \leq \frac{(p+1)^3}{2p^2(p+2)} & (3.2) \\ \left[\frac{4p^4\mu}{(p+1)^2} - \frac{p^2(2p+1)}{p+2}\right]; \ if \ \mu \geq \frac{(p+1)^3}{2p^2(p+2)} & (3.3) \end{cases}$$

Proof: By definition of $f(z) \in KS_p^*$, , we have$\frac{z(zf'(z))'}{z(f'(z))} = p\frac{1+w(z)}{1-w(z)}$; $w(z) \in \mathcal{U}$. (3.4)

Expanding the series (3.4), we get

$$\{p^2 z^p + (p+1)^2 a_{p+1} z^{p+1} + (p+2)^2 a_{p+2} z^{p+2} - -\} = p\,(1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + --)(3.5)$$

Identifying terms in (3.5), we get

$$a_{p+1} = \frac{2p^2}{(p+1)} c_1 (3.6)$$

$$a_{p+1} = \frac{p^2}{(p+2)} c_2 + \frac{[p^2(2p+1))]}{(p+2)} {c_1}^2 .(3.7)$$

From (3.6) and (3.7), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{(p)^2}{(p+2)} c_2 + \left[\frac{2p^3+p^2}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right] c_1^2 (3.8)$$

Taking absolute value, it can be rewritten as

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)}|c_2| + \left|\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right| |c_1^2|. \quad (3.9)$$

Using (1.11) in (3.9), we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)}(1 - |c_1|^2) + \left|\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right| |c_1^2|$$

$$= \frac{(p)^2}{(p+2)} + \left[\left|\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right| - \frac{(p)^2}{(p+2)}\right] |c_1|^2 .(3.10)$$

Case I:$\mu \leq \frac{(p+1)^2(2p+1)}{4p^2(p+2)}$. (3.10) can be rewritten as

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} + \left[\frac{2(p)^3}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right] |c_1|^2.$$

Subcase I (a): $\mu \leq \frac{(p+1)^2}{2p(p+2)}$. Using (1.11), (3.11) becomes

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \left[\frac{p^2(2p+1)}{(p+2)} - \frac{4p^4\mu}{(p+1)^2}\right] (3.12)$$

Subcase I (b): $\mu \geq \frac{(p+1)^2}{2p(p+2)}$. We obtain from (3.11)

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} .(3.13)$$

Case II: $\mu \geq \frac{(p+1)^2(2p+1)}{4p^2(p+2)}$

Preceding as in case I, we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} + \left[\frac{4p^4\mu}{(p+1)^2} - \frac{p^2(p+1)}{(p+2)}\right] |c_1|^2 .(3.14)$$

Subcase II (a): $\mu \leq \frac{(p+1)^3}{2p^2(p+2)}$

(3.14) takes the form

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} \quad (3.15)$$

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{(p)^2}{(p+2)} \;;\; if \; \frac{p^2(2p+1)}{(p+2)} \leq \mu \leq \frac{(p+1)^3}{2p^2(p+2)} \quad (3.16)$$

Subcase II (b): $\mu \geq \frac{(p+1)^3}{2p^2(p+2)}$

Preceding as in subcase I (a), we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \left[\frac{4p^4\mu}{(p+1)^2} - \frac{p^2(2p+1)}{(p+2)}\right] \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = \frac{z^p}{p} + \frac{p\, z^{p+1}}{p+1} + \frac{p(p-1)}{2!} \quad \frac{z^{p+2}}{p+2} + - - - - - -$$

Extremal function for (3.2) is defined by $f_2(z) = \frac{z^p}{p} + \frac{p\, z^{p+2}}{p+2} + \frac{p(p-1)}{2!} \quad \frac{z^{p+4}}{p+4} + - - - - - -.$

Corollary 3.2: Putting $p = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, if \mu \leq \frac{2}{3}; \\ \frac{1}{3} if \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \mu - 1, if \; \mu \geq \frac{4}{3} \end{cases}$$

These are the required results of class KS*.

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