Forward-Backward Splitting Method with Viscosity Iteration technique for Solving Monotone Inclusion Problems

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Abstract- In this paper, we introduce and study a modified forward-backward splitting method for finding a zero in the sum of two monotone operators in real Hilbert spaces. Our proposed method only requires one forward evaluation of the single-valued operator and one backward evaluation of the set-valued operator per iteration. This is an improvement over many others in literature with strongly convergent splitting methods with two forwards and a backward iteration. Furthermore, we also incorporate inertial term in our scheme to speed up the rate of convergence. We obtain a strong convergence result when the set-valued operator is maximal monotone and the single-valued operator is Lipschitz continuous monotone which is weaker assumption than being inverse strongly monotone or cocoercive.

Indexed Terms- Viscosity Iteration Method; Inertial Method; Inclusion Problem; Maximal Monotone Operator; Forward–Backward Algorithm.

I. INTRODUCTION

Let H be a real Hilbert space with an induced norm $\|.\|$ and inner product $\langle .,. \rangle$.

The monotone inclusion problem (MIP) is defined as follows:

$$find\ x \in H\ such\ that$$
$$0 \in (A+B)x, \qquad (1.1)$$

where $A: H \to H$ and $B: H \to 2^H$ are monotone operator. Let the solution set of (1.1) be denoted by $(A+B)^{-1}(0)$. The inclusions of the form specified in (1.1) arises in numerous real-world problems. It plays a central role in mathematical optimizations such as variational inequalities, minimization problems, linear inverse problems, saddle-point problems, fixed point problems, split feasibility problems, Nash equilibrium problems in noncooperative games, and many more (see [1] and references contained therein). In what follows, we provide motivating examples.

- Convex minimization. A minimization problem has the following structure:

$$\min_{x \in H} f(x) + g(x),$$

where $f: \mathcal{H} \to (-\infty, +\infty]$ is proper, lower semicontinuous (lsc), convex and $f: \mathcal{H} \to \mathbb{R}$ is convex with (locally) Lipschitz continuous gradient denoted by ∇g. It is well known that the solutions to this minimization problem are those points in H which satisfiy the first order optimality condition:

$$0 \in (\partial f + \nabla g)x, \qquad (1.2)$$

where ∂f is the subdifferential of f. It is not hard to see that (1.2) is clearly of the form of (1.1).

- General monotone inclusions. Consider the inclusion problem

$$find\ x \in H\ such\ that$$
$$0 \in (A + K^*BK)(x), \qquad (1.3)$$

where $A: H_1 \rightrightarrows H_1$ and $B: H_2 \rightrightarrows H_2$ are maximally monotone operators, and $K: H_1 \to H_2$ is abounded linear operator and K^* its adjoint. It was noted in [2,3] that solving (1.3) is equivalent to solving the following inclusion problem:

$$find\ \begin{pmatrix} x \\ y \end{pmatrix} \in H_1 \times H_2$$
$$such\ that\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \left(\begin{bmatrix} A & 0 \\ 0 & B^{-1} \end{bmatrix} + \begin{bmatrix} 0 & K^* \\ -K & 0 \end{bmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix}. \qquad (1.4)$$

It is also clear that (1.4) is of the form (1.1).

- Saddle point problems and variational inequalities. Many convex optimization problems can be formulated as the saddle point problem

$$\min_{x \in H} \max_{y \in H} g(x) + \Phi(x,y) - f(y), \qquad (1.5)$$

where $f, g: H \to (-\infty, +\infty]$ are proper, lsc, convex functions and $\Phi: \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is smooth convex-concave function. The problem (1.5) naturally arises in machine learning, statistics among others where the dual (maximization) problem comes either from daulizing the constrains in the primal problem or from using the Fenchel-Legendre transform to leverage a nonsmooth composite part. Following the first order optimality condition, the problem (1.5) can be expressed as the monotone inclusion

$$find \binom{x}{y} \in H \times H \ such \ that \binom{0}{0} \in \binom{\partial g(x)}{\partial f(y)} + \binom{\nabla_x \Phi(x,y)}{\nabla_y \Phi(x,y)}, \quad (1.6)$$

which is of the form specified in (1.1). Using the definition of the sub-differentials, equation (1.6) can be expressed in the form of the variational inequality:

$$\begin{cases} find\ z^* = (x^*, y^*)^T \in H \times H\ such\ that \\ \langle B(z^*), z - z^* \rangle + g(x) - g(x^*) \qquad (1.7) \\ -f(y) + f(y^*) \geq 0, \forall z = \binom{x}{y} \in H \times H \end{cases}$$

where $(x, y) \coloneqq (\nabla_x \Phi(x,y) - \nabla_x \Phi(x,y))^T$. Therefore, inclusion problems are primarily of natural interest for those studying pure and applied sciences.

However, monotone operator theory is a fascinating field of research in nonlinear functional analysis and found valuable applications in the field of convex optimization, subgradients, partial differential equations, variational inequalities, signal and image processing, evolution equations and inclusions; see, for instance, [4,5,6] and the references cited therein.

There are many methods for solving (1.1) among them is the splitting algorithm which is defined in the following iterative step: $given\ x_1 \in H$ and

$$x_{n+1} = (I + rA)^{-1}(x_n - rBx_n) \qquad (1.8)$$

where $r \in (0, \frac{0}{L})$, $J_r^A = (I + rA)$ is called the resolvent of A and $I: H \to H$ denotes the identity operator. The iterative sequence $\{x_n\}$ converges weakly to a solution provided the operator $B: H \to B$ is $1/L$- cocoercive. By being cocoercive, we mean that $\langle x - y, B(x) - B(y) \rangle \geq \beta \|B(x) - B(y)\|^2 \ \forall x, y \in H$.

In this iterative form (1.8), the individual steps within each iterations involves forward evaluations in which the value of the single-valued operator is computed and the backward evaluations in which the resolvent of the set-valued operator is computed rather than their sum directly. Many researchers have constructed different algorithms using (1.8) (see for e.g., [4,8,9] and references therein).

The cocoercive property of the operator B is a stronger than Lipschitz continuity. Therefore, it is difficult to satisfy general monotone inclusions. In order to relax this assumption, Tseng [7] proposed a modification of (1.8) known as the Tseng's method or the known forward-backward-forward method. This method require two forward evaluations with the operator B being Lipchitz continuous. In fact, [7] presented the following iterative method:

$$\begin{cases} y_n = J_r^A(x_n - rBx_n), \\ x_{n+1} = y_n - rBy_n + rBx_n, \qquad (1.9) \\ \qquad \forall\, n \in \mathbb{N}, \end{cases}$$

and a weak convergence result was obtained under the assumption that B is L −Lipschitz and $r \in \left(0, \frac{2}{L}\right)$.

In 2001, Alvarez and Attouch [10] used the concept of heavy ball method which was equally studied by [11, 12] to study maximal monotone operators by the proximal point algorithm. They presented the following algorithm called inertial proximal point algorithm:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n B)^{-1} y_n, \qquad (1.10) \\ \qquad \forall\, n \in \mathbb{N}, \end{cases}$$

Under some mild conditions which include the fact that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, $\theta_n \in [0, 1)$ and $\{r_n\}$ is nondecreasing. They prove that $\{x_n\}$ converged weakly to the solution set. Here θ_n is called the inertial factor while $\theta_n(x_n - x_{n-1})$ is called the term. It has been proven that the inertial terms often speeds up the rate of convergence (see [13,14,15]).

It is desirable in applications to obtain strong convergence results of proposed algorithms. In 2018, Khan et al. [16] used the shrinking method developed by Takahashi et al. [17] to establish a strong convergence for monotone inclusion problem. They also used the inertial term to ensure fast convergence. In fact, in [16] the following algorithm is presented:$x_0, x_1 \in C = H$,

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n x_n + (1-\alpha_n)J_{r_n}^B(I - r_n A)y_n, \\ C_{n+1} = \begin{Bmatrix} z \in C_n: \|z_n - z\|^2 \le \|x_n - z\|^2 \\ +2\theta_n^2\|x_n - x_{n-1}\|^2 \\ -2\theta_n(1-\alpha_n)\langle x_n - z, x_n - x_{n-1}\rangle \end{Bmatrix} \quad (1.11) \\ x_{n+1} = P_{C_{n+1}}x_1, \ n \ge 1, \end{cases}$$

under some mild conditions see [16, Theorem 3.1], they proved that the sequence $\{x_n\}$ converged strongly to the solution set.

- Remark 1.12: We noticed that the algorithm (1.11) of Khan et al (2018) has the following drawbacks: i)

a. The single-valued operator is α −inverse strongly monotone which is the same as being cocoercive. This condition is stronger than being Lipschitz continuous and monotone.

b. It is very natural to have slow convergence withii) projection operators. Hence, the algorithm (1.11) due to projection onto closed and convex setiii) $\{C_{n+1}\}$ has slow convergence irrespective of the fact that the inertial term is incorporated.

- The natural question to ask is: can these conditions be relaxed and obtain a strong convergence? It is our purpose in this paper to give an affirmative answer to the above question.

The rest of the paper is organized as follows: in section 2, we present some basic definitions very relevant to our work and as well state without proofs of vital Lemmas. Our algorithm is presented in section 3 with the conditions that would ensure strong convergence and the proof. In section 4, we provide a conclusion of all we have done.

II. PRELIMINARY

Let C be a nonempty closed convex subset of a real Hilbert space . The weak and strong convergence of a sequence $\{x_n\}$ is denoted by $x_n \rightharpoonup x$ and $x_n \longrightarrow x$, respectively. For each $x, y, z \in H$, the following facts are known:

- $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;
- $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \alpha \in \Re$;

For every point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $P_C(x) := argmin\{\|x-y\|, y \in C\}$, where P_C is called the metric projection of $\mathcal{H}$ onto C. It is well known that P_C is nonexpansive and $P_C(x)$ is characterized by:

a. $\langle x - P_C(x), y - P_C(x)\rangle \le 0, \forall x \in H, y \in C$;
b. $\|P_C(x) - P_C(y)\|^2 \le \langle P_C(x) - P_C(y), x - y\rangle, \forall x, y \in \mathcal{H}$.

The following definition shall be very useful in our work. For any $x, y \in \mathcal{H}$, the mapping $T: \mathcal{H} \to \mathcal{H}$ is said to be:

L −Lipschitz continuous with $L > 0$ if

$$\|Tx - Ty\| \le L\|x - y\|.$$

If $L \in (0,1)$, then the mapping T is called a contraction map. In particular, if $L = 1$, the mapping T is called a nonexpansive map.

Monotone if

$$\langle Tx - Ty, x - y\rangle \ge 0.$$

α −inverse strongly monotone (also called α −cocoercive) if

$\langle Tx - Ty, x - y\rangle$
$\ge \alpha\|Tx - Ty\|^2$.

We observe from the above definitions that every α −inverse strongly monotone is $\frac{1}{\alpha}$ −Lipschitz continuous and monotone.

A multi-valued mapping $A: \mathcal{H} \to 2^{\mathcal{H}}$ is called:

a. Monotone if $\langle u - v, x - y\rangle \ge 0, \forall x, y \in \mathcal{H}$ whenever $u \in Ax$ and $v \in Ay$. $Dom(A) = \{x \in \mathcal{H}: Ax \ne 0\}$.
b. Maximal monotone if it is monotone and if for any $(x, u) \in \mathcal{H} \times \mathcal{H}, \langle u - v, x - y\rangle \ge 0$ for every $(y, v) \in Graph(A)$.
c. The resolvent of operator associated with maximal monotone A with a positive number λ is denoted by J_λ^A and it is defined on $\mathcal{H}$ by $J_\lambda^A(x) = (I + \lambda A)^{-1}$.

We shall state the following Lemmas without their proofs. Readers who are interested for their proofs can consult the materials.

- Lemma 2.1 [18]: Assume that $\mathcal{H}$ is a real Hilbert space. Let the mapping $A: \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous monotone and mapping $B: \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone. Then, the mapping $(A+B)$ is a maximal monotone.

- Lemma 2.2 [19]: Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in (0,1) with $\sum_{n=1}^{\infty}\alpha_n=\infty$, and $\{b_n\}$ be sequence of real numbers. Assume that

$$a_{n+1}\le(1-\alpha_n)+\alpha_n b_n,\qquad n\ge 1.$$

If $\limsup_{k\to\infty} b_{n_k}\le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\lim_{k\to\infty}\inf(a_{n_k+1}-a_{n_k})\ge 0$, then $\lim_{n\to\infty}a_n=0$.

III. THE MAIN RESULT

In this section, we state our algorithm and state some assumptions that will enable us to establish a strong convergence.

(C1) The solution set $(A+B)^{-1}\neq\emptyset$.

(C2)The mapping $A:\mathcal{H}\to\mathcal{H}$ is L −Lipschitz continuous and monotone and $B:\mathcal{H}\to 2^{\mathcal{H}}$ is maximal monotone.

(C3) The mapping $f:\mathcal{H}\to\mathcal{H}$ is a contraction mapping with a coefficient $\rho\in[0,1)$.

(C4) Let $\{\varepsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_n}=0$, where $\alpha_n\in(0,1)$ satisfies $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$.

Algorithm 3.1: The inertial viscosity iteration scheme for monotone inclusion problems

Initialization: Set $\lambda_0\ge 0,\mu\in(0,1)$ and $x_0,x_1\in\mathcal{H}$ be arbitrary.

Iterative steps. Step 1: Given the current iterates x_{n-1}, $and\ x_n$ $(n\ge 1)$, choose θ_n such that $0\le\theta_n\le\overline{\theta_n}$, where

$$\overline{\theta_n}:=\begin{cases}\min\left\{\frac{\varepsilon_n}{\|x_n-x_{n-1}\|}\right\} & ,if\, x_n\neq x_{n-1};\\ \theta, & otherwise\,.\end{cases}\qquad(3.2)$$

Step 2: Compute

$$\begin{cases}w_n=x_n+\theta_n(x_n-x_{n-1}),\\ y_n=J_{\lambda_n}^{B}(I-\lambda_n A)w_n,\\ z_n=y_n-\lambda_n(Ay_n-Ax_n),\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)z_n,\end{cases}\qquad(3.3)$$

Where $\lambda_{n+1}=$

$$\begin{cases}\min\left\{\frac{\mu\|w_n-y_n\|}{\|Aw_n-Ay_n\|},\lambda_n\right\}, if\ Aw_n\neq Ay_n,\\ \lambda_n & otherwise,\end{cases}\qquad(3.4)$$

Set $n:=n+1$ and go to Step 1.

Lemma 3.2: The sequence $\{\lambda_n\}$ formed by (3.4) is nonincreasing and $\lim_{n\to\infty}\lambda_n=\lambda\ge\min\{\lambda_0,\frac{\mu}{L}\}$.

Proof: It is not hard to see the nonincreasing of $\{\lambda_n\}$ from its definition. Moreover, from the fact that the operator A is L −Lipschitz continuous, we get

$$\frac{\mu\|w_n-y_n\|}{\|Aw_n-Ay_n\|}\ge\frac{\mu}{L}, if\ Aw_n\neq Ay_n.$$

Thus, we deduce that $\lim_{n\to\infty}\lambda_n=\lambda\ge\min\{\lambda_0,\frac{\mu}{L}\}$. This completes the proof. ■

Lemma 3.3: If the conditions (C1)-(C4) are satisfied, then

$$\|z_n-p\|^2\le\|w_n-p\|^2-\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n-y_n\|^2, p\in\Omega.$$

Proof: From the definition of z_n in (3.3) and for all $p\in\Omega$, we get

$$\begin{aligned}\|z_n-p\|^2&=\|y_n-\lambda_n(Aw_n-Ay_n)-p\|^2\\&=\|y_n-p\|^2+\lambda_n^2\|Aw_n-Ay_n\|^2-2\lambda_n\langle y_n-p,Aw_n-Ay_n\rangle\\&=\|w_n-p\|^2+\|y_n-w_n\|^2-2\lambda_n\langle y_n-p,Aw_n-Ay_n\rangle+2\langle y_n-w_n,y_n-p\rangle+\lambda_n^2\|Aw_n-Ay_n\|^2\qquad(3.5)\\&=\|w_n-p\|^2-\|w_n-y_n\|^2-2\langle w_n-y_n-\lambda_n(Aw_n-Ay_n),y_n-p\rangle\end{aligned}$$

Now, using the definition of λ_n in (3.4), we obtain

$$\|Aw_n-Ay_n\|\le\frac{\mu}{\lambda_{n+1}}\|w_n-y_n\|, \text{for all n.}\qquad(3.6)$$

We observe that the above inequality holds even when $Aw_n=Ay_n$. Otherwise, we obtain

$$\lambda_{n+1}\le\frac{\mu\|w_n-y_n\|}{\|Aw_n-Ay_n\|},$$

which yields that $\|Aw_n-Ay_n\|\le\frac{\mu}{\lambda_{n+1}}\|w_n-y_n\|$.

Therefore, the inequality holds for all n

Next, $\langle w_n-y_n-\lambda_n(Aw_n-Ay_n),y_n-p\rangle\ge 0$.

Using the definition of y_n from the algorithm, we get that

$$(1-\lambda_n A)w_n\in(I+\lambda_n B)y_n.$$

Using the maximal monotonicity of B, there exists $v_n\in By_n$ such that

$$(1-\lambda_n A)w_n=y_n+\lambda_n v_n.$$

This indicates that

$$v_n=\frac{1}{\lambda_n}(w_n-y_n-\lambda_n Aw_n)\qquad(3.7)$$

Now, for all $p \in \Omega$, we have $0 \in (A+B)(p)$. From $Ay_n + v_n \in (A+B)y_n$ and the fact that $(A+B)$ is a maximal monotone (see Lemma 2.1), we have $\langle Ay_n + v_n, y_n - p\rangle \geq 0$. This together with (3.7) gives

$$\lambda_n^{-1}\langle w_n - y_n - \lambda_n A w_n + \lambda_n A y_n, y_n - p\rangle \geq 0,$$

which further implies that (3.7) holds.

Now combining (3.5)-(3.7), we get

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2.$$ This completes the proof. ■

We now state and proof the main Theorem of this paper.

Theorem 3.4: Suppose that the Assumptions (C1)-(C4) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $p \in \Omega$ in norm, where $\|p\| = min\{\|z\| : z \in \Omega\}$.

Proof: We first show that $\{x_n\}$ is bounded.

By the definition of w_n and for all $p \in \Omega$, we get

$$\|w_n - p\| \leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \quad (3.8)$$

We know from the definition of θ_n that

$$\theta_n\|x_n - x_{n-1}\| \leq \varepsilon_n, \forall n.$$

Therefore,

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\varepsilon_n}{\alpha_n} \to 0 \; as \; n \to \infty.$$

Hence, there exists a constant $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1, \forall\, n \geq 1. \quad (3.9)$$

So, (3.8) can be written as

$$\|w_n - p\| = \|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \quad (3.10)$$
$$\leq \|x_n - p\| + \alpha_n M_1, \quad \forall\, n \geq 1.$$

From the fact that

$$\lim_{n\to\infty}\left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \mu^2 > 0,$$

we get that there exists $n_0 \in \mathbb{N}$ such that

$$\left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) > 0 \;\forall\, n \geq n_0. \quad (3.11)$$

Combining (3.9) –(3.10)and Lemma 3.3, we get

$$\|z_n - p\| \leq \|x_n - p\| + \alpha_n M_1. \quad (3.12)$$

Next, we compute, using the estimate in (3.12)

$$\|x_{n+1} - p\| = \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(z_n - p)\|$$
$$\leq \alpha_n\|f(x_n) - p\| + (1-\alpha_n)\|z_n - p\|$$
$$= \alpha_n\|f(x_n) - f(p) + f(p) - p\| + (1-\alpha_n)\|z_n - p\|$$
$$\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|z_n - p\|$$
$$\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|w_n - p\|$$
$$\leq \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)[\|x_n - p\| + \alpha_n M_1]$$
$$= [\alpha_n\rho + (1-\alpha_n)]\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\alpha_n M_1$$
$$\leq [\alpha_n\rho + (1-\alpha_n)]\|x_n - p\| + \alpha_n\|f(p) - p\| + \alpha_n M_1$$
$$= \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\| + \alpha_n(1-\rho)\left(\frac{\|f(p)-p\| + M_1}{(1-\rho)}\right) \quad (3.13)$$
$$\leq \max\left\{\|x_n - p\|, \frac{\|f(p) - p\| + M_1}{(1-\rho)}\right\}$$
$$\leq \cdots \leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\| + M_1}{(1-\rho)}\right\}$$

This implies that the sequence $\{x_n\}$ is bounded. It follows that the sequence $\{w_n\}, \{z_n\}, \{y_n\}$ and $\{f(x_n)\}$ are all bounded.

From (3.10), we get that

$$\|w_n - p\|^2 \leq (\|x_n - p\| + \alpha_n M_1)^2$$
$$= \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \quad (3.14)$$
$$\leq \|x_n - p\|^2 + \alpha_n M_2 \text{ for some } M_2 > 0.$$

Combining Lemma 2.2 and (3.14)

$$\|x_{n+1} - p\|^2 \leq \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|)^2 + (1-\alpha_n)\|z_n - p\|^2$$
$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + \alpha_n(2\|x_n - p\|.\|f(p) - p\| + \|f(p) - p\|^2)$$
$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + \alpha_n M_3$$
$$\leq \|x_n - p\|^2 - (1-\alpha_n)\left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 + \alpha_n M_4$$

where $M_4 := M_2 + M_3$. That is,

$$(1-\alpha_n)\left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4 \quad (3.15)$$

In view of definition w_n, we get

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2$$
$$\leq \|x_n - p\|^2 + 3M\theta_n\|x_n - x_{n-1}\|,$$

where $M := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \theta_n\|x_n - x_{n-1}\|\} > 0$.

We further estimate that

$$\|x_{n+1}-p\|^2 = \|\alpha_n(f(x_n)-f(p)) + (1-\alpha_n)(z_n-p) + \alpha_n(f(p)-p)\|^2$$
$$\leq \|\alpha_n(f(x_n)-f(p)) + (1-\alpha_n)(z_n-p)\|^2$$
$$+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle$$
$$\leq \alpha_n\|f(x_n)-f(p)\|^2 + (1-\alpha_n)\|z_n-p\|^2$$
$$+\alpha_n\langle f(p)-p, x_{n+1}-p\rangle$$
$$\leq \alpha_n\rho\|x_n-p\|^2 + (1-\alpha_n)\|w_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle$$
$$\leq (1-(1-\rho)\alpha_n)\|x_n-p\|^2 + (1-\rho)\alpha_n\left[\frac{3M}{(1-\rho)}.\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + \frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle\right] \quad (3.16)$$

Finally, we show that $\{\|x_n-p\|^2\}$ converges to zero. We may assume without loss of generality that $\{\|x_{n_k}-p\|\}$ is a subsequence of $\{\|x_n-p\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-p\| - \|x_{n_k}-p\|) \geq 0$.

By the condition on (C3) and (3.15), one gets

$$(1-\alpha_{n_k})\left(1-\mu^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\right)\|w_{n_k}-y_{n_k}\|^2 \leq \limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2 - \|x_{n_k+1}-p\|^2 + \alpha_{n_k}M_4\right]$$

which implies that

$$\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\| = 0. \quad (3.17)$$

Using the definition of z_n and (3.6), we deduce that

$$\|z_n-y_n\| \leq \mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n-y_n\|,$$

this yields that

$$\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\| = 0. \quad (1.18)$$

Using (3.17) and (3.18), wet get

$$\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\| \leq \lim_{k\to\infty}(\|w_{n_k}-y_{n_k}\| + \|y_{n_k}-z_{n_k}\|) = 0. \quad (3.19)$$

Furthermore, from the Algorithm and the condition on (C4) one gets

$$\|x_{n_k+1}-z_{n_k}\| = \alpha_{n_k}\|z_{n_k}-f(x_{n_k})\| \to 0 \quad (3.20)$$

and

$$\|w_{n_k}-x_{n_k}\| = \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\| \to 0 \quad (3.21)$$

From (3.20) and (3.21), we obtain

$$\|x_{n_k+1}-x_{n_k}\| \leq \|x_{n_k+1}-z_{n_k}\| + \|z_{n_k}-w_{n_k}\| + \|w_{n_k}-x_{n_k}\| \quad (3.22)$$

It follows from (3.22) that

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\| = 0. \quad (3.23)$$

Since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$ as $j \longrightarrow \infty$.

Furthermore,

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \lim_{j\to\infty}\langle f(p)-p, x_{n_{k_j}}-p\rangle = \langle f(p)-p, z-p\rangle \quad (3.24)$$

However, we also obtain that $w_{n_k} \rightharpoonup z$ as $j \longrightarrow \infty$ since $\|w_{n_k}-x_{n_k}\| \to 0$. This together with $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\| = 0$ and Lemma 2.2 give that $z \in \Omega$. Now, using the definition of p and the fact in (3.24), we get

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \langle f(p)-p, z-p\rangle \leq 0. \quad (3.25)$$

Combining (3.23) and (3.25), one gets

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k+1}-p\rangle \leq \limsup_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle \leq 0. \quad (3.26)$$

Using (3.20)-(3.26), we deduce that

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k+1}-p\rangle \leq 0.$$

This together with Lemma 2.2 and (3.16), we conclude that $x_n \to p$. This completes the proof■

CONCLUSION

Using viscosity iterative method that involved an inertial term, a strong convergence was obtained under mild assumptions. Our scheme involved one forward and one backward per iteration. More so, we relaxed the strongly inverse monotone assumption on one of the operators and adopted Lipschtiz continuity and monotone. Our computation is simple and easily implemented. Our result is an improvement to many results in the literature

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