Transitivity of the Product Action of Finite Alternating Groups on Cartesian Product of Finite Ordered Sets 0f γ-Tuples

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Abstract- In this paper, we determine the transitivity of the product action of finite alternating groups on the Cartesian product of finite ordered sets of γ-tuples. Transitivity action has been determined using the Orbit-stabilizer theorem, by showing that the length of the orbit $(p_1, p_2, p_3, \dots, p_{m-1}, p_m)$ in $A_{n_1} \times A_{n_2} \times \dots \times A_{n_{m-1}} \times A_{n_m}$, $(n - \gamma \geq 2)$ acting on $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times \dots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$ is equivalent to the cardinality of $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times \dots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$ to imply transitivity.

Indexed Terms- Orbits; stabilizer; transitive group; ordered sets of γ-tuples; cartesian product; fixed point.

I. INTRODUCTION

The transitivity of the product action of two, three and four alternating groups, that is, $A_{n_1} \times A_{n_2}$, $A_{n_1} \times A_{n_2} \times A_{n_3}$ and , $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$ on the Cartesian product of two, three and four sets of ordered γ-tuples respectively have first been determined hence, forming the basis for the generalization (main theorem) for the transitivity of the product action of finite alternating groups, $A_{n_1} \times A_{n_2} \times \dots \times A_{n_{m-1}} \times A_{n_m}$, on the Cartesian product of finite sets ordered γ-tuples, $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times \dots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$.
Given the sets $P_1, P_2, \dots, P_{m-1}$ and P_m such that $P_1 = \{1,2,3,\dots,n\}$, $P_2 = \{n+1, n+2, n+3, \dots, 2n\}$,…………, $P_{m-1} = \{(m-2)n+1, (m-2)n+2, \dots, (m-1)n-1, (m-1)n\}$ and

$P_m = \{(m-1)n+1, (m-1)n+2, \dots, (mn)-2, (mn)-1, mn\}$. Then, the sets of ordered $\gamma - tuples$ from these sets are: $\{[1,2,3,\dots,\gamma], [1,2,3,\dots,\gamma+1], \dots, [n, n-1, n-2, \dots, n-\gamma+1]\} \in P_1{}^{[\gamma]}$, $\{[n+1, n+2, \dots, n+\gamma], [n+1, n+2, \dots, n+\gamma+1], \dots, [2n, 2n-1, 2n-2, \dots, 2n-\gamma+1]\} \in P_2{}^{[\gamma]}$,
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$\{[(m-2)n+1, (m-2)n+2, \dots, (m-2)n+\gamma], [(m-2)n+1, (m-2)n+2, \dots, (m-2)n+\gamma+1], \dots, [(m-1)n, (m-1)n-1, (m-1)n-2 \dots, (m-1)n-\gamma+1]\} \in P_{m-1}{}^{[\gamma]}$ and

$\{[(m-1)n+1, (m-1)n+2, \dots, (m-1)n+\gamma], [(m-1)n+1, (m-1)n+2, \dots, (m-1)n+\gamma+1], \dots, [mn, mn-1, mn-2 \dots, mn-\gamma+1]\} \in P_m{}^{[\gamma]}$ respectively.

The Cartesian product of sets of ordered $\gamma - tuples$ $P_1{}^{[\gamma]}$, $P_2{}^{[\gamma]}, \dots, P_{m-1}{}^{[\gamma]}$ and $P_m{}^{[\gamma]}$ is defined as the set of all $\gamma - tuples$ $(p_1, p_2, p_3, \dots, p_{m-1}, p_m)$ such that p_1 belongs to $P_1{}^{[\gamma]}$, p_2 belongs to $P_2{}^{[\gamma]}$, p_3 belongs to $P_3{}^{[\gamma]}$,….., p_{m-1} belongs to $P_{m-1}{}^{[\gamma]}$ and p_m belongs to $P_m{}^{[\gamma]}$.

Higman (1964) introduced the rank of a group on finite permutation groups of rank 3. In 1970, Higman proved that the rank of the symmetric group S_n acting on 2-element subsets from the set P= $\{1,2,\dots,n\}$ is 3 and the subdegrees are:1,$2(n-1)$ and $\binom{n-2}{2}$.

Ndarinyo *et al* (2015) showed that the alternating group A_n acts transitively on unordered and ordered triples from the set P= $1,2,\dots,n$ when $n \leq 7$. Nyaga (2018) proved that the direct product action of the alternating group on the Cartesian product of three sets is transitive. Mutua *et. al.*, (2018): direct product of $S_n \times A_n$ on P× S has its action both transitive and imprimitive when $n \geq 3$.

Maraka *et al.*, (2021) showed that the action of the cartesian product of the alternating group, $A_n \times A_n \times A_n$, on the cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$, the cartesian product of ordered sets of triples is transitive when $n \geq 5$.

Maraka *et al.*, (2022) showed that the action of the cartesian product of the alternating group, $A_n \times A_n \times A_n$, on the cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$, the cartesian product of ordered sets of triples is imprimitive when $n \geq 5$.

Definition 1.1. Group action (Njagi, 2016)

Given a group G and a non-empty set P, the action of G to the left of P matches a unique element $gp \in P$ if $\forall\, g \in G$ such that for all $p \in P$ and $g_1, g_2 \in G$:

(i) $(g_1 g_2)p = g_1 (g_2)p$.
(ii) $\varrho . p = p$, given that ϱ is the identity in G.
When G acts from the right side of P, its action can similarly be denoted as such.

Definition 1.2. Transitive group (Cameron, 1970)

A group G is termed to act transitively on a set P provided for all $p, s \in P$, $\exists g \in G: g(p) = s$, that is, the action gives only a single orbit.

Definition 1.3. Stabilizer of an Element (Rose, 1978)

Let $p \in$ P and a group G act on P. The stabilizer of p in G is given by $Stab_G(p) = \{g \,\epsilon G: gp = p\}$.

Definition 1.4. Fixed point (Njagi, 2016)

Given a non-empty set P and group G acting on P with $g \in G$. The set of elements p fixed by $g \in G$ is referred to as fixed point set of h given by $Fix\,(g) = \{p \,\epsilon P: h(p) = p\}$.

Theorem 1.5. Orbit – Stabilizer Theorem (Rose, 1978)

Given G acts on a set P, $|Orb_G\,(p)| = |G: Stab_G(p)|$.

II. MAIN RESULTS

2.1 Transitivity of the Product Action of Two Alternating Groups, $A_{n_1} \times A_{n_2}$, on Cartesian Products of Two Sets of Ordered γ -Tuples $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$

2.1 Order of the Stabilizer, $Stab_G([1,2,3, \ldots, \gamma], [n+1, n+2, \ldots, n+\gamma])$

Let $G = A_{n_1} \times A_{n_2}$ and $d_1 = Stab_G([1,2,3, \ldots, \gamma], [n+1, n+2, \ldots, n+\gamma])\_$ be the stabilizer of $\{[1,2,3, \ldots, \gamma], [n+1, n+2, \ldots, n+\gamma]\} \in P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$ in G, then the following result follows.

Proposition 2.1

$|d_1| = |Stab_G([1,2,3, \ldots, \gamma], [n+1, n+2, \ldots, n+\gamma])| = \left(\frac{(n-\gamma)!}{2}\right)^2$.

Proof: Let $([1,2,3, \ldots, \gamma], [n+1, n+2, \ldots, n+\gamma]) \in P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$ and $g_1, g_2 \in G$. Then, g_1 fixes $[1,2,3, \ldots, \gamma]$ if and only if each of the elements $1,2,3, \ldots, \gamma-1$ and γ comes from a single cycle in g_1 and g_2 fixes $[n+1, n+2, \ldots, n+\gamma]$ if and only if each of the elements $n+1, n+2, \ldots, n+\gamma-1$ and $n+\gamma$ comes from a single cycle in g_2.

Hence, the order of the stabilizer, $Stab_G([1,2,3, \ldots, \gamma], [n+1, n+2, \ldots, n+\gamma])$, is equal to the order of the group of all permutations of the set $\{[1,2,3, \ldots, \gamma+1], [n+1, n+2, \ldots, n+\gamma+1]\}$. But this group is isomorphic to the Cartesian product of the alternating groups; $A_{n_1 - }\gamma \times A_{n_2 - }\gamma$. Thus, $|d_1| = \frac{(n_1-\gamma)\times(n_2-\gamma)}{2_1 \times 2_2}$.

But, $A_{n_1} = A_{n_2} = A_n$, so, $n_1 = n_2 = n$.

Therefore, $|d_1| = \frac{(n-\gamma)\times(n-\gamma)}{2\times 2} = \frac{((n-\gamma)!)^2}{2^2}$.

$|d_1| = \left(\frac{(n-\gamma)!}{2}\right)^2$.

2.2 Transitivity of the Product Action of Two Alternating Groups, $A_{n_1} \times A_{n_2}$, on Cartesian Products of Two Sets of Ordered γ -Tuples $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$

Proposition 2.2

The product action of two alternating groups, $A_{n_1} \times A_{n_2}$, acts transitively on the Cartesian product of two sets of ordered γ –tuples, $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$ if and only if $n-\gamma \geq 2$.

Proof:

Let $G = G_{P_1} \times G_{P_2} = A_{n_1} \times A_{n_2}$ act on $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$.

The cardinality of $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$ is given as;

$$\left|P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}\right| = n_{P_\gamma} \times n_{P_\gamma}$$

$$= \frac{n!}{(n-\gamma)!} \times \frac{n!}{(n-\gamma)!} = \left(\frac{n!}{(n-\gamma)!}\right)^2$$

It suffices to show that $\left|P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}\right|$ is equal to $|Orb_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma])|$.

Let $|d_1| = |Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma])|$.

So, $(g_{P_1}, g_{P_2}) \in G = A_{n_1} \times A_{n_2}$ fixes $([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma])$ if and only if $1,2,3,\dots,\gamma-1$ and γ comes from a $1-$cycle of g_{P_1} and $n+1,n+2,\dots,n+\gamma-1$ and $n+\gamma$ comes from a $1-$cycle of g_{P_2}.

From Theorem 1.5, we get;

$$|Orb_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma])| = |G: Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma])|$$

$|G| = \frac{n!\times n!}{2\times 2} = \left(\frac{n!}{2}\right)^2$.

$\frac{|G|}{|d_1|} = \frac{\left(\frac{n!}{2}\right)^2}{\left(\frac{(n-\gamma)!}{2}\right)^2}$.

Therefore, $\frac{|G|}{|d_1|} = \left(\frac{n!}{(n-\gamma)!}\right)^2 = \left|P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}\right|$.

When $n-\gamma \leq 2$, $|d_1| = \left|A_{n-\gamma} \times A_{n-\gamma}\right| < 1$.

This implies that, $|d_1| = \left(\frac{(n-\gamma)!}{2}\right)^2 < 1$, $\forall\, n-\gamma \leq 2$ yet $|d_1| \in Z^+$.

Hence, $A_{n_1} \times A_{n_2}$ acts transitively on $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]}$ if and only if $n-\gamma \geq 2$.

2.3. Transitivity of the Product Action of Three Alternating Groups, $A_{n_1} \times A_{n_2} \times A_{n_3}$, on Cartesian Products of Three Sets of Ordered γ -Tuples $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]}$

2.3 Order of the Stabilizer, $Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])$

Let $G = A_{n_1} \times A_{n_2} \times A_{n_3}$ and $d_2 = Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])$ be the stabilizer of $\{[1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma]\} \in P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]}$ in G, then the following result follows.

Proposition 2.3

$|d_2| = |Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])| = \left(\frac{(n-\gamma)!}{2}\right)^3$.

Proof: Let $([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma]) \in P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]}$ and $g_1, g_2, g_3 \in G$. Then, g_1 fixes $[1,2,3,\dots,\gamma]$ if and only if each of the elements $1,2,3,\dots,\gamma-1$ and γ comes from a single cycle in g_1; g_2 fixes $[n+1,n+2,\dots,n+\gamma]$ if and only if each of the elements $n+1,n+2,\dots,n+\gamma-1$ and $n+\gamma$ comes from a single cycle in g_2 and g_3 fixes $[2n+1,2n+2,\dots,2n+\gamma]$ if and only if each of the elements $2n+1,2n+2,\dots,2n+\gamma-1$ and $2n+\gamma$ comes from a single cycle in g_3

Therefore, the order of the stabilizer, $Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])$, is equal to the order of the group of all permutations of the set $\{[1,2,3,\dots,\gamma+1],[n+1,n+2,\dots,n+\gamma+1],[2n+1,2n+2,\dots,2n+\gamma+1]\}$ which is isomorphic to the Cartesian product of the alternating groups; $A_{n_1-}\gamma \times A_{n_2-}\gamma \times A_{n_3-}\gamma$.

Thus, $|\boldsymbol{d_2}| = \frac{(\boldsymbol{n_1}-\boldsymbol{\gamma})\times(\boldsymbol{n_2}-\boldsymbol{\gamma})\times(\boldsymbol{n_3}-\boldsymbol{\gamma})}{\boldsymbol{2_1}\times\boldsymbol{2_2}\times\boldsymbol{2_3}}$.

But, $A_{n_1} = A_{n_2} = A_{n_3} = A_n$, so, $n_1 = n_2 = n_3 = n$.

So, $|\boldsymbol{d_2}| = \frac{(\boldsymbol{n}-\boldsymbol{\gamma})\times(\boldsymbol{n}-\boldsymbol{\gamma})\times(\boldsymbol{n}-\boldsymbol{\gamma})}{\boldsymbol{2}\times\boldsymbol{2}\times\boldsymbol{2}} = \frac{((\boldsymbol{n}-\boldsymbol{\gamma})!)^{\boldsymbol{3}}}{\boldsymbol{2^3}}$.

Therefore, $|\boldsymbol{d_2}| = \left(\frac{(\boldsymbol{n}-\boldsymbol{\gamma})!}{\boldsymbol{2}}\right)^{\boldsymbol{3}}$.

2.4 Transitivity of the Product Action of Three Alternating Groups, $A_{n_1} \times A_{n_2} \times A_{n_3}$, on Cartesian Products of Three Sets of Ordered γ -Tuples ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}$

Proposition 2.4

The product action of three alternating groups, $A_{n_1} \times A_{n_2} \times A_{n_3}$, acts transitively on the Cartesian product of three sets of ordered γ –tuples, ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}$ if and only if $n-\gamma \geq 2$.

Proof:

Let $G = G_{P_1} \times G_{P_2} \times G_{P_3} = A_{n_1} \times A_{n_2} \times A_{n_3}$ act on ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}$.

The cardinality of ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}$ is given as;

$$\left|{P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}\right| = n_{P_\gamma} \times n_{P_\gamma} \times n_{P_\gamma}$$

$$= \frac{n!}{(n-\gamma)!} \times \frac{n!}{(n-\gamma)!} \times \frac{n!}{(n-\gamma)!} = \left(\frac{n!}{(n-\gamma)!}\right)^3$$

It suffices to show that $\left|{P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}\right|$ is equal to $|Orb_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])|$.

Let $|d_2| = |Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])|$.

So, $(g_{P_1}, g_{P_2}, g_{P_3}) \in G = A_{n_1} \times A_{n_2} \times A_{n_3}$ fixes $([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])$ if and only if $1,2,3,\dots,\gamma-1$ and γ comes from a $1-$cycle of g_{P_1}; $n+1,n+2,\dots,n+\gamma-1$ and $n+\gamma$ comes from a $1-$cycle of g_{P_2} and $2n+1,2n+2,\dots,2n+\gamma-1$ and $2n+\gamma$ comes from a $1-$cycle of g_{P_3}.

From Theorem 1.5, we get;

$$|Orb_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])| = |G: Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma])|$$

$|G| = \frac{n!\times n!\times n!}{2\times2\times2} = \left(\frac{n!}{2}\right)^3$.

$\frac{|G|}{|d_2|} = \frac{\left(\frac{n!}{2}\right)^3}{\left(\frac{(n-\gamma)!}{2}\right)^3}$.

Therefore, $\frac{|G|}{|d_2|} = \left(\frac{n!}{(n-\gamma)!}\right)^3 = \left|{P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}\right|$.

When $n-\gamma \leq 2$, $|d_2| = \left|A_{n-\gamma} \times A_{n-\gamma} \times A_{n-\gamma}\right| < 1$.

This implies that, $|d_2| = \left(\frac{(n-\gamma)!}{2}\right)^3 < 1$, $\forall\, n-\gamma \leq 2$ yet $|d_2| \in Z^+$.

Hence, $A_{n_1} \times A_{n_2} \times A_{n_3}$, acts transitively on ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]}$ if and only if $n-\gamma \geq 2$.

2.5. Transitivity of the Product Action of Four Alternating Groups, $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$, on Cartesian Products of Four Sets of Ordered γ -Tuples ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]} \times {P_4}^{[\gamma]}$

2.5 Order of the Stabilizer, $Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])$

Let $G = A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$, and $d_3 = Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])$ be the stabilizer of $\{[1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma]\} \in {P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times {P_3}^{[\gamma]} \times {P_4}^{[\gamma]}$ in G, then the following result follows.

Proposition 2.5

$|d_3| = |Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])| = \left(\frac{(n-\gamma)!}{2}\right)^4$.

Proof: Let $([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+$

$2,\dots,3n+\gamma]) \in P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}$ and $g_1, g_2, g_3, g_4 \in G$. Then, g_1 fixes $[1,2,3,\dots,\gamma]$ if and only if each of the elements $1,2,3,\dots,\gamma-1$ and γ comes from a single cycle in g_1; g_2 fixes $[n+1, n+2,\dots,n+\gamma]$ if and only if each of the elements $n+1, n+2,\dots,n+\gamma-1$ and $n+\gamma$ comes from a single cycle in g_2; g_3 fixes $[2n+1,2n+2,\dots,2n+\gamma]$ if and only if each of the elements $2n+1,2n+2,\dots,2n+\gamma-1$ and $2n+\gamma$ comes from a single cycle in g_3 and g_4 fixes $[3n+1,3n+2,\dots,3n+\gamma]$ if and only if each of the elements $3n+1,3n+2,\dots,3n+\gamma-1$ and $3n+\gamma$ comes from a single cycle in g_4.

Therefore, the order of the stabilizer, $Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])$, is equal to the order of the group of all permutations of the set $\{[1,2,3,\dots,\gamma+1],[n+1,n+2,\dots,n+\gamma+1],[2n+1,2n+2,\dots,2n+\gamma+1],[3n+1,3n+2,\dots,3n+\gamma+1]\}$ which is isomorphic to the Cartesian product of the alternating groups; $A_{n_1-}\gamma \times A_{n_2-}\gamma \times A_{n_3-}\gamma \times A_{n_4-}\gamma$.

Thus, $|d_3| = \frac{(n_1-\gamma)\times(n_2-\gamma)\times(n_3-\gamma)\times(n_4-\gamma)}{2_1\times 2_2\times 2_3\times 2_4}$.

But, $A_{n_1} = A_{n_2} = A_{n_3} = A_{n_4} = A_n$, so, $n_1 = n_2 = n_3 = n_4 = n$.

So, $|d_3| = \frac{(n-\gamma)\times(n-\gamma)\times(n-\gamma)\times(n-\gamma)}{2\times2\times2\times2} = \frac{((n-\gamma)!)^4}{2^4}$.

Therefore, $|d_3| = \left(\frac{(n-\gamma)!}{2}\right)^4$.

2.6 Transitivity of the Product Action of Four Alternating Groups, $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$, on Cartesian Products of Four Sets of Ordered γ -Tuples $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}$

Proposition 2.6

The product action of alternating groups, $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$, acts transitively on the Cartesian product of four sets of ordered γ –tuples, $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}$ if and only if $n-\gamma \geq 2$.

Proof:

Let $G = G_{P_1} \times G_{P_2} \times G_{P_3} \times G_{P_4} = A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$ act on $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}$.

The cardinality of $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}$ is given as;

$$\left|P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}\right| = n_{P_\gamma} \times n_{P_\gamma} \times n_{P_\gamma} \times n_{P_\gamma}$$

$$= \frac{n!}{(n-\gamma)!} \times \frac{n!}{(n-\gamma)!} \times \frac{n!}{(n-\gamma)!} \times \frac{n!}{(n-\gamma)!}$$

$$= \left(\frac{n!}{(n-\gamma)!}\right)^4$$

It suffices to show that $\left|P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}\right|$ is equal to $|Orb_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])|$.

Let $|d_3| = |Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])|$.

So, $(g_{P_1}, g_{P_2}, g_{P_3}, g_{P_4}) \in G = A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$ fixes $([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])$ if and only if $1,2,3,\dots,\gamma-1$ and γ comes from a $1-$cycle of g_{P_1}; $n+1,n+2,\dots,n+\gamma-1$ and $n+\gamma$ comes from a $1-$cycle of g_{P_2}; $2n+1,2n+2,\dots,2n+\gamma-1$ and $2n+\gamma$ comes from a $1-$cycle of g_{P_3} and $3n+1,3n+2,\dots,3n+\gamma-1$ and $3n+\gamma$ comes from a single cycle of g_{P_4}.

From Theorem 1.5, we get;

$$|Orb_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])| = |G: Stab_G([1,2,3,\dots,\gamma],[n+1,n+2,\dots,n+\gamma],[2n+1,2n+2,\dots,2n+\gamma],[3n+1,3n+2,\dots,3n+\gamma])|$$

$$|G| = \frac{n!\times n!\times n!\times n!}{2\times2\times2\times2} = \left(\frac{n!}{2}\right)^4$$

$\frac{|G|}{|d_3|} = \frac{\left(\frac{n!}{2}\right)^4}{\left(\frac{(n-\gamma)!}{2}\right)^4}$.

Therefore, $\frac{|G|}{|d_3|} = \left(\frac{n!}{(n-\gamma)!}\right)^4 = \left|P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}\right|$.

When $n-\gamma \le 2$, $|d_3| = \left|A_{n-\gamma} \times A_{n-\gamma} \times A_{n-\gamma} \times A_{n-\gamma}\right| < 1$.

This implies that, $|d_3| = \left(\frac{(n-\gamma)!}{2}\right)^4 < 1,\ \forall\, n-\gamma \le 2$ yet $|d_3| \in Z^+$.

Hence, $A_{n_1} \times A_{n_2} \times A_{n_3} \times A_{n_4}$, acts transitively on $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times P_3{}^{[\gamma]} \times P_4{}^{[\gamma]}$ if and only if $n-\gamma \ge 2$.

Remark: The proof for Theorem 2.7 and 2.8 follows from Propositions 2.2, 2.4 and 2.6.

2.7. Transitivity of the Product Action of Finite Alternating Groups, $A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$, on Cartesian Product of Finite Sets of Ordered γ -Tuples, $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times \ldots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$

2.7. Order of the Stabilizer, $Stab_G([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],[2n+1,2n+2,\ldots,2n+\gamma],[3n+1,3n+2,\ldots,3n+\gamma],\ldots,[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])$

Let $G = A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$, and $d_m = Stab_G([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],[2n+1,2n+2,\ldots,2n+\gamma],[3n+1,3n+2,\ldots,3n+\gamma],\ldots,[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])$ be the stabilizer of $\{[1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],[2n+1,2n+2,\ldots,2n+\gamma],[3n+1,3n+2,\ldots,3n+\gamma],\ldots,[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma]\} \in P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \ldots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$ in G, then the following result follows.

Theorem 2.7

$|d_m| = |Stab_G([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],[2n+1,2n+2,\ldots,2n+\gamma],[3n+1,3n+2,\ldots,3n+\gamma],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])| = \left(\frac{(n-\gamma)!}{2}\right)^m$.

Proof: Let $([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],[2n+1,2n+2,\ldots,2n+\gamma],[3n+1,3n+2,\ldots,3n+\gamma],\ldots,[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma]) \in P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \ldots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$ and $g_1, g_2, g_3, \ldots, g_{m-1}, g_m \in G$. Then, g_1 fixes $[1,2,3,\ldots,\gamma]$ if and only if each of the elements $1,2,3,\ldots,\gamma-1$ and γ comes from a single cycle of g_1; g_2 fixes $[n+1,n+2,\ldots,n+\gamma]$ if and only if each of the elements $n+1,n+2,\ldots,n+\gamma-1$ and $n+\gamma$ comes from a single cycle of g_2; g_3 fixes $[2n+1,2n+2,\ldots,2n+\gamma]$ if and only if each of the elements $2n+1,2n+2,\ldots,2n+\gamma-1$ and $2n+\gamma$ comes from a single cycle of g_3 ; g_4 fixes $[3n+1,3n+2,\ldots,3n+\gamma]$ if and only if each of the elements $3n+1,3n+2,\ldots,3n+\gamma-1$ and $3n+\gamma$ comes from a single cycle of g_4;………….., g_{m-1} fixes $[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma]$ if and only if each of the elements $(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma-1$ and $(m-2)n+\gamma$ comes from a single cycle in g_{m-1} and g_m fixes $[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma]$ if and only if each of the elements $(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma-1$ and $(m-1)n+\gamma$ comes from a single cycle of g_m.

Therefore, the order of the stabilizer, $Stab_G([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])$, is equal to the order of the group of all permutations of the set $\{[1,2,3,\ldots,\gamma+1],[n+1,n+2,\ldots,n+\gamma+1],[2n+1,2n+2,\ldots,2n+\gamma+1],[3n+1,3n+2,\ldots,3n+\gamma+1],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma+1],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma+1]\}$ which is isomorphic to the Cartesian product of the alternating groups; $A_{n_1-\gamma} \times A_{n_2-\gamma} \times A_{n_3-\gamma} \times A_{n_4-\gamma} \times\ldots\times A_{n_{m-1}-\gamma} \times A_{n_m-\gamma}$.

Therefore, $|d_m| = \frac{(n_1-\gamma)\times(n_2-\gamma)\times(n_3-\gamma)\times(n_4-\gamma)\times\ldots\times(n_{m-1}-\gamma)\times(n_m-\gamma)}{2_1\times 2_2\times 2_3\times 2_4\times\ldots\times 2_{m-1}\times 2_m}$.

But, $A_{n_1} = A_{n_2} = A_{n_3} = A_{n_4} = \cdots = A_{n_{m-1}} = A_{n_m} = A_n$, so, $n_1 = n_2 = n_3 = n_4 = \cdots = n_{m-1} = n_m = n$.

Accordingly, $2_1 = 2_2 = 2_3 = 2_4 = \cdots = 2_{m-1} = 2_m = 2$.

Thus, $|d_m| = \frac{(n_1-\gamma)\times(n_2-\gamma)\times(n_3-\gamma)\times(n_4-\gamma)\times\ldots\times(n_{m-1}-\gamma)\times(n_m-\gamma)}{2^m} = \frac{((n-\gamma)!)^m}{2^m}$.

Hence, $|d_m| = \left(\frac{(n-\gamma)!}{2}\right)^m$.

2.8 Transitivity of the Product Action of Finite Alternating Groups, $A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$, on Cartesian Products of Finite Sets of Ordered γ -Tuples, ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times \ldots \times {P_{m-1}}^{[\gamma]} \times {P_m}^{[\gamma]}$

Theorem 2.8

The product action of finite alternating groups, $A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$, acts transitively on the Cartesian product of finite sets of ordered γ – tuples, ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times \ldots \times {P_{m-1}}^{[\gamma]} \times {P_m}^{[\gamma]}$ if and only if $n-\gamma \geq 2$.

Proof:

The proof is done in a similar manner as in Propositions 2.2, 2.4 and 2.6.

Let $G = G_{P_1} \times G_{P_2} \times \ldots \times G_{P_{m-1}} \times G_{P_m} = A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$ act on ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times \ldots \times {P_{m-1}}^{[\gamma]} \times {P_m}^{[\gamma]}$.

The cardinality of ${P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times \ldots \times {P_{m-1}}^{[\gamma]} \times {P_m}^{[\gamma]}$ is given as;

$$\left|{P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times \ldots \times {P_{m-1}}^{[\gamma]} \times {P_m}^{[\gamma]}\right| = n_{{P_1}_\gamma} \times n_{{P_2}_\gamma} \times \ldots \times n_{{P_{m-1}}_\gamma} \times n_{{P_m}_\gamma}$$

$$= \frac{n_1!}{(n_1-\gamma)!} \times \frac{n_2!}{(n_2-\gamma)!} \times \ldots \times \frac{n_{m-1}!}{(n_{m-1}-\gamma)!} \times \frac{n_m!}{(n_m-\gamma)!} = \left(\frac{n!}{(n-\gamma)!}\right)^m$$

It suffices to show that $\left|{P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times \ldots \times {P_{m-1}}^{[\gamma]} \times {P_m}^{[\gamma]}\right|$ is equal to $|Orb_G([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],[2n+1,2n+2,\ldots,2n+\gamma],[3n+1,3n+2,\ldots,3n+\gamma],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])|$.

So, $(g_{P_1}, g_{P_2}, \ldots, g_{P_{m-1}}, g_{P_m}) \in G = A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$ fixes $([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],[2n+1,2n+2,\ldots,2n+\gamma],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])$ if and only if $1,2,3,\ldots,\gamma-1$ and γ comes from a $1-$cycle of g_{P_1}; $n+1,n+2,\ldots,n+\gamma-1$ and $n+\gamma$ comes from a $1-$cycle of g_{P_2}; $2n+1,2n+2,\ldots,2n+\gamma-1$ and $2n+\gamma$ comes from a $1-$cycle of g_{P_3}, ………….., $(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma-1$ and $(m-2)n+\gamma$ comes from a single cycle of $g_{P_{m-1}}$ and $(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma-1$ and $(m-1)n+\gamma$ comes from a single cycle of g_{P_m}.

From Theorem 2.7, the order of the stabilizer, $Stab_G([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])$ is given as;

$|d_m| = \left(\frac{(n-\gamma)!}{2}\right)^m$.

Applying Theorem 1.5, we get;

$$\begin{aligned}|Orb_G([1,2,3,\ldots,\gamma],&[n+1,n+2,\ldots,n+\gamma],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])| \\ &= |G{:}Stab_G([1,2,3,\ldots,\gamma],[n+1,n+2,\ldots,n+\gamma],\ldots,[(m-2)n+1,(m-2)n+2,\ldots,(m-2)n+\gamma],[(m-1)n+1,(m-1)n+2,\ldots,(m-1)n+\gamma])|\end{aligned}$$

$$|G| = \frac{n_1! \times n_2! \times \ldots \times n_{m-1}! \times n_m!}{2_1 \times 2_2 \times \ldots \times 2_{m-1} \times 2_m} = \left(\frac{n!}{2}\right)^m$$

$\frac{|G|}{|d_m|} = \frac{\left(\frac{n!}{2}\right)^m}{\left(\frac{(n-\gamma)!}{2}\right)^m}$.

Therefore, $\frac{|G|}{|d_m|} = \left(\frac{n!}{(n-\gamma)!}\right)^m = \left|{P_1}^{[\gamma]} \times {P_2}^{[\gamma]} \times \ldots \times {P_{m-1}}^{[\gamma]} \times {P_m}^{[\gamma]}\right|$.

When $n-\gamma \leq 2$, $|d_m| = \left|A_{n_1-\gamma} \times A_{n_2-\gamma} \times \ldots \times A_{n_{m-1}-\gamma} \times A_{n_m-\gamma}\right| < 1$.

This implies that, $|d_m| = \left(\frac{(n-\gamma)!}{2}\right)^m < 1,\ \forall\, n-\gamma \leq 2$ yet $|d_m| \in Z^+$.

Hence, $A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$, acts transitively on $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times \ldots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$ if and only if $n-\gamma \geq 2$.

CONCLUSION

The product action of finite alternating groups acts transitively on the cartesian product of finite sets of ordered $\gamma - tuples$, that is $A_{n_1} \times A_{n_2} \times \ldots \times A_{n_{m-1}} \times A_{n_m}$ on $P_1{}^{[\gamma]} \times P_2{}^{[\gamma]} \times \ldots \times P_{m-1}{}^{[\gamma]} \times P_m{}^{[\gamma]}$, if $n-\gamma \geq 2$.

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