Higher Order Perturbative Calculations of Nuclear Finite-Size Effects on The Energy Levels of Exotic Atoms

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Abstract- ***This research explores the nuclear finite-size effects on the energy levels of exotic atoms, focusing on theoretical calculations up to the fourth order. The study employs a robust framework to derive and analyze energy-level shifts influenced by the nuclear charge distribution for selected atomic numbers (Z = 1, 3, 11, 19, 37, 55, 87). The theoretical models account for higher-order corrections, providing a systematic improvement in accuracy compared to experimental results. The findings highlight that second-order calculations yield reasonable approximations, but significant deviations are observed, particularly at higher atomic numbers. Third-order calculations exhibit improved agreement but still deviate in precision for larger nuclei. Fourth-order corrections achieve remarkable accuracy, with deviations from experimental values becoming negligible. Graphical analyses further illustrate the trends and convergence of the theoretical models to experimental data.***

Indexed Terms- ***Nuclear, finite size, perturbation Exotic, energy, Brillion-wegner, muonic, model, experiment, charge distribution, hifht-order, atomic number. Frame work, nuclear-structure***

I. INTRODUCTION

Exotic atoms, which consist of a negatively charged particle (such as a muon or antiproton) bound to a nucleus, offer a unique probe of nuclear and atomic properties. The study of exotic atoms has long been a vibrant area of research, with recent advances in experimental techniques and theoretical methodologies opening up new avenues for exploration. One important aspect of exotic atom Physics is the role of nuclear finite-size effects, which can significantly impact the energy levels and other properties of these systems.

Simple atoms play an important role in the check of quantum electrodynamics, the bound state theory and precise determination of fundamental physical constants (the fine structure constant, the lepton and proton masses, the Rydberg constant, the proton charge radius, etc.). However, for simple atoms, the changes in electron energy states due to nuclear structure effects, are very small or not observed at all (Elekina et. al., 2011). On the other hand, the simplest and at the same time exotic atomic system is the highly charged hydrogen-like muonic atom and ions, where a negative muon μ^- is bound to an atomic nucleus (Patoary and Oreshkina, 2018). This particular class of atoms offers an interesting opportunity to extract nuclear structure effects with high accuracy (Antognini, 2015). The previous investigation on nuclear structure effects (Adamu and Ngadda, 2014; Adamu and Ngadda, 2015; Palffy, 2011; El Shabshiry et. al., 2015) showed that such effects are very small and can be ignored for light nucleus and on p, d, f, … energy states. It is suggested that these effects are more important for higher atomic nucleus and muonic atoms. Being 207 times heavier than an electron (m_μ = 207 m_e), and distinctly small atomic orbitals radii, muonic atoms have an enhanced sensitivity to nuclear structure effects (Kanda, 2022; Godunov and Vysotsky, 2013; Antognini, 2015; Dong et. al., 2011; Toth et. al., 2021; Antognini et. al., 2022). Other nuclear parameters such as root-mean-square radii, electric quadrupole and magnetic dipole moments were also extracted based on theoretical predictions and experimental measurements of the level structure and the transition energies in muonic atoms (Patoary and Oreshkina, 2018; Faustov et. al., 2019).

First-order perturbative calculations of nuclear finite-size effects on exotic atom energy levels are insufficient, and higher-order corrections are necessary to achieve precise agreement with experimental data. Exact calculations of nuclear finite-

size effects on exotic atom energy levels are computationally prohibitive, and an efficient perturbative approach is needed to enable calculations for a wide range of exotic atoms. The absence of a systematic perturbative framework for nuclear finite-size effects on exotic atom energy levels hinders the development of a comprehensive understanding of these effects across different energy states.

The research aims to improve the accuracy of theoretical predictions for muonic atom energy levels, enabling better agreement with experimental data and deeper understanding of atomic physics and By exploring second-order perturbative corrections, the research will provide new insights into the physical mechanisms underlying nuclear finite-size effects, shedding light on the intricate relationships between nuclear structure and atomic properties.

Quantum mechanics is one of the most peculiar theories that gives numerical results agree extremely well with experiments (Thaller, 2005; pp 57). It successfully expressed the internal electronic structure of atoms on the basis of general physical law. One of the quantum mechanical equations is Schrödinger's equation, which has been applied to theoretically explain the spectrum of an atom when the lone electron is solely subjected to Z/r interaction (Landau and Lifshitz, 1991; Herzberg, 2014; White, 1934; Sala et. al., 1999; Rae, 2008). However, the energy spectrum, obtained from Schrödinger equation, resulted in a higher number of degenerate states than the observed spectra (El-Shabshiry et. al., 2013; Landau and Lifshitz, 1991; Schrodinger, 1930; Palffy, 2011; White, 1934; Sala et. al., 1999). The reason behind these discrepancies is in the fact that the Z/r potential energy that described the nuclear-lepton interaction is only an approximation, as it does not take into account the fact that, atomic electrons are sensitive to other interactions of the nucleus they are bound to, such as nuclear mass, charge distribution, spin or its magnetization distribution. These nuclear properties affect the nuclear-lepton interaction and give rise a Hamiltonians in which the total potential is a sum of an exactly solvable potential plus a weak, or small potential and the solution to this new Hamiltonian are sought approximately. Therefore, this new Hamiltonian cannot be described by Schrödinger equation as they totally affect the Z/r interaction and thus the Hamiltonian (Greiner, 2001; Adamu and Ngadda, 2017). Due to complex structure of nucleus, an orbiting lepton interacts not only with $+Ze$ nuclear charge but with other nuclear properties that give rise to Hamiltonian in which the total potential is a sum of an exact solvable Z/r potential plus a weak or small perturbation (Schiff, 1949; Greiner, 2001). Any physical phenomenon that involves the nucleus of atom, falls into the category of quantum mechanical changes and must be observed by approximation methods. It is therefore proven essential to develop simple and adequate approximation method that can provide approximate solution to the eigenvalue equation of the perturbed Hamiltonian. Approximation methods played an important part in virtually all applications of the theory and the widely used approximation method in atomic, nuclear and particle physics is the perturbation theory. Perturbation theory plays an important role in virtually all application in the theory in atomic, nuclear and particle physics. It has been the backbone to the interpretation of spectroscopic changes and in most cases, the time independent perturbation theory was applied to determine some nuclear structure effects (Schiff, 1949; Gasiorovicz, 2003; Newton, 2002) which consequently, resulted in the fine structure, hyperfine splittings respectively and Lamb Shifts. Quantum electrodynamics theory provides a better description of the system by adding radiative effects that take into account the interaction of an electron with fluctuating vacuum fields and vacuum polarization (Thaller, 1992). Thus the known features of atomic spectra have been at least semi-quantitatively explained in terms of the nuclear atom, strictly according to quantum mechanical laws and quantum electrodynamics theory (Evans, 2015; Thaller, 2005).

While first-order perturbation theory has been widely used to study nuclear finite-size effects in exotic atoms, second-order perturbation theory has received less attention. However, second-order perturbation theory can provide a more accurate and comprehensive understanding of these effects, particularly for systems with strong nuclear potentials. In this work, we present a theoretical study of nuclear finite-size effects on the energy levels of exotic atoms using second-order perturbation theory. Our calculations aim to elucidate the importance of

second-order perturbative corrections and provide a more detailed understanding of the interplay between nuclear and atomic properties in exotic atoms.

II. METHODOLOGY

2.1 Model Formulation

The research applied mathematical methods to estimate or expand what have been reviewed in the approximation methods. The research applied quantum electrodynamics theory and Brillouin – Wigner time independent perturbation and obtain equations that can best describe the finite-size model. The research also employed numerical techniques using computer program (Microsoft Excel).

2.2 The Modified Nuclear Interaction

The new lepton-nuclear interaction that describes the interior of the nucleus of charge $+Ze$ as a uniformly charged sphere of radius R and a spherical Gauss surface of radius r as shown in Figure 2.1.

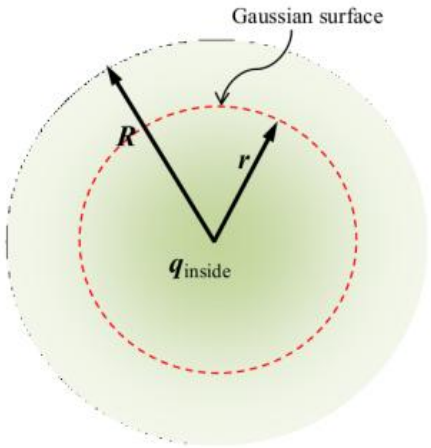


Fig. 2.1: The Gaussian surface inside the charge distribution

From this figure, for $r < R$, the total charge inside a sphere of radius r is

$$q_{\text{inside}} = +Ze\left(\frac{r}{R}\right)^3 \qquad (2.1)$$

To determine the internal and external electric fields and hence the electrostatic potential, ϕ, we apply Gauss' law which states:

$$\iint \vec{E}\cdot d\vec{s} = \frac{q_{inside}}{\varepsilon_0}$$

By symmetry, the electric field, E is purely radial and so the flux through the Gaussian surface is $E \times 4\pi r^2$. Since E is constant, then

$$4\pi r^2 E = \frac{q_{inside}}{\varepsilon_0} = \frac{+Ze}{\varepsilon_0}\left(\frac{r}{R}\right)^3$$

and therefore,

$$E = \frac{Zer}{4\pi\varepsilon_0 R^3} = -\frac{d\phi}{dr}$$

Or

$$\phi = -\frac{Ze}{4\pi\varepsilon_0 R^3}\int r\,dr$$

$$= -\frac{Zke}{R^3}\frac{r^2}{2}$$

$$+ C \qquad (2.2)$$

where C is a constant of integration. In a region $r > R$, the electric potential from Coulomb's law states:

$$\phi = \frac{+Zke}{r} \qquad (2.3)$$

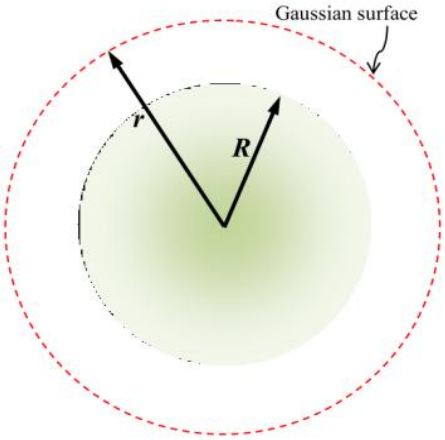


Fig. 2.2: The Gaussian surface outside the charge distribution

The electric field inside a sphere of radius r (Figure 1.2) is the same as for a point charge e, located at the origin. By matching the interior (1.2) and exterior (2.2) solutions for ϕ at the surface of the nucleus, $r = R$ we find:

$$C = \frac{3Zke}{2R}$$

Thus, for $r \leq R$ we have from (2.2) that, the electrostatic potential:

$$\phi = \frac{Zke}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right] \qquad (2.4)$$

and the corresponding expression for the potential energy is described by

$$U(\vec{r}, R) = -\frac{\gamma}{2r}\left[\frac{3r}{R} - \left(\frac{r}{R}\right)^3\right] \qquad (2.5)$$

where $\gamma = Zke^2$is the coulomb constant and within a nuclear radius $r \leq R$, $R = r_0A^{1/3}$, with $r_0 = 1.2 \times 10^{-15}m$. Equation 3.4 represents the potential for a finite-size charge nucleus.

2.3 The Brillouin-Wigner Time Independent Perturbation Theory

In section 3.1 the research has developed a new nuclear interaction (2.4) due to the finite size of atomic nucleus. To solve for such interaction, the new (perturbed) Hamiltonian can be written as the sum of an exactly solvable potential plus a weak, or small, potential as follows:

$$\hat{H}_{\text{pert.}} = \hat{H}_0 + \lambda\hat{H}$$

where

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\nabla^2 - \frac{\gamma}{r}$$

and the perturbation,

$$\lambda\hat{H} = \frac{\gamma}{R}\left[\frac{R}{r} - \frac{3}{R} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \qquad (2.6)$$

The parameter $\lambda\hat{H}'$ is small enough so that it can be required as a perturbation on $\hat{H}_0$ and λ is a real parameter which allows the expansion of wave functions and energies into a power series in λ. The parameter λ is also called the smallness (or perturbation) parameter and is taken to be dimensionless small number, $\lambda \ll 1$, so that the perturbation $\lambda\hat{H}'$ is small compared to the original Hamiltonian $\hat{H}_0$ (Greiner, 2001). The most widely used approximation techniques for solving the Schrödinger equation in such a special situation (2.18) is the Perturbation theory. Its mathematical techniques are among the most widely used in nuclear and atomic physics (Greenbaum et. al., 2020; (Messiah, 1976; Landau and Lifshitz, 2004).

Thus substitution yield the following:

$$\mathcal{E}_n = \hat{E}_0 + \lambda h_{nn} + \lambda^2 \frac{h_{nm}h_{mn}}{(\hat{E}'_n - \hat{E}_m)} + \lambda^3 \frac{h_{nm}h_{ml}h_{ln}}{(\hat{E}'_n - \hat{E}_m)(\hat{E} - \hat{E}_l)} + \lambda^4 \frac{h_{nm}h_{ml}h_{lk}h_{kn}}{(\hat{E}'_n - \hat{E}_m)(\hat{E}'_n - \hat{E}_l)(\hat{E}'_n - \hat{E}_k)} + \cdots \qquad (2.7)$$

The projection operator automatically excludes certain matrix elements from the summations in each term of the series. This formula is very similar to the Rayleigh-Schrödinger series of conventional perturbation theory, except that the perturbed energy $\hat{E}'_n$ appears in each denominator instead of the unperturbed energy $\hat{E}_n$. One can prove that the Brillouin–Wigner expansion does not contain terms in which the state $|\psi_m\rangle$ appears in the numerator as an intermediate state. The last term in (2.28) appears as a third-order term in the Raleigh–Schrödinger perturbation theory (2.30), does not exist in the Brillouin–Wigner expansion, since it has been taken into account through the replacement in the denominator of the second-order term (2.30). However, the advantage of reducing the number of terms may be compensated by a decrease in the convergence of the perturbation expansion, associated with the nature of the partial summations. Moreover, different powers of $\hat{E}_1$ may be present in many terms of the Brillouin–Wigner series (Bes, 2012).

III. RESULTS AND DISCUSSION

RESULTS

Table 1. 1st to 4th order energy shifts for the wave function E_{100} for hydrogenic atoms

n	Z	1ST ORDER	2ND ORDER	3RD ORDER	4TH ORDER
1	1	0.04948594	8.698910^{-07}	5.4317610^{-15}	8.305810^{-26}
1	1	0.07500666	1.998510^{-06}	1.891510^{-14}	1.007210^{-24}
1	1	0.09566533	3.25110^{-06}	3.9244610^{-14}	4.335710^{-24}
1	3	1.30501367	6.72210^{-05}	1.3666510^{-13}	4.73210^{-22}
1	3	1.43147292	8.08810^{-05}	1.8037110^{-13}	8.242610^{-22}
1	3	1.55088039	9.493610^{-05}	2.2938210^{-13}	1.33310^{-21}
1	3	1.66444641	0.00010935	2.8355710^{-13}	2.037110^{-21}

1	3	1.87741344	0.00013912	4.0693110^{-13}	4.19531^{-21}
1	11	39.2817254	0.0045304	1.534041^{-12}	8.01531^{-19}
1	11	39.2919764	0.00453277	1.535251^{-12}	8.02791^{-19}
1	11	40.3082476	0.00477029	1.657491^{-12}	9.35721^{-19}
1	19	161.169081	0.02556324	3.990271^{-12}	1.61791^{-17}
1	19	163.389822	0.02627262	4.157531^{-12}	1.75641^{-17}
1	37	960.134568	0.23925428	1.547331^{-11}	9.22511^{-16}
1	37	973.949874	0.24618992	1.615111^{-11}	1.00511^{-15}
1	37	980.808726	0.24967007	1.649481^{-11}	1.04831^{-15}
1	55	2814.11458	0.93023453	3.612041^{-11}	1.11071^{-14}
1	55	2815.25777	0.9309906	3.616441^{-11}	1.11341^{-14}
1	55	2827.93915	0.93939821	3.665551^{-11}	1.14381^{-14}
1	87	9317.98108	4.07648497	8.373211^{-11}	1.49331^{-13}
1	87	9553.34962	4.28507413	9.024191^{-11}	1.73451^{-13}
1	87	9605.12952	4.33166142	9.171781^{-11}	1.79171^{-13}
2	1	0.00309287	5.43681^{-08}	5.431761^{-15}	5.19111^{-27}
2	1	0.00468792	1.24911^{-07}	1.89151^{-14}	6.29491^{-26}
2	1	0.00597908	2.03191^{-07}	3.924461^{-14}	2.70981^{-25}
2	3	0.08156335	4.20131^{-06}	1.366651^{-13}	2.95751^{-23}
2	3	0.08946706	5.0551^{-06}	1.803711^{-13}	5.15161^{-23}
2	3	0.09693002	5.93351^{-06}	2.293821^{-13}	8.33151^{-23}
2	3	0.1040279	6.83431^{-06}	2.835571^{-13}	1.27321^{-22}
2	3	0.11733834	8.69521^{-06}	4.069311^{-13}	2.62211^{-22}
2	11	2.45510784	0.00028315	1.534041^{-12}	5.00961^{-20}
2	11	2.45574852	0.0002833	1.535251^{-12}	5.01741^{-20}
2	11	2.51926548	0.00029814	1.657491^{-12}	5.84831^{-20}
2	19	10.0730676	0.0015977	3.990271^{-12}	1.01121^{-18}
2	19	10.2118639	0.00164204	4.157531^{-12}	1.09771^{-18}
2	37	60.0084105	0.01495339	1.547331^{-11}	5.76571^{-17}
2	37	60.8718671	0.01538687	1.615111^{-11}	6.28181^{-17}
2	37	61.3005454	0.01560438	1.649481^{-11}	6.5521^{-17}
2	55	175.882161	0.05813966	3.612041^{-11}	6.94181^{-16}
2	55	175.953611	0.05818691	3.616441^{-11}	6.95881^{-16}
2	55	176.746197	0.05871239	3.665551^{-11}	7.14911^{-16}
2	87	582.373817	0.25478031	8.373211^{-11}	$9.333E1^{-15}$
2	87	597.084351	0.26781713	9.024191^{-11}	1.0841^{-14}
2	87	600.320595	0.27072884	9.171781^{-11}	1.11981^{-14}
3	1	0.00061094	1.07391^{-08}	5.431761^{-15}	1.02541^{-27}
3	1	0.00092601	2.46731^{-08}	1.89151^{-14}	1.24341^{-26}
3	1	0.00118105	4.01361^{-08}	3.924461^{-14}	5.35271^{-26}
3	3	0.01611128	8.29881^{-07}	1.366651^{-13}	5.8421^{-24}
3	3	0.01767251	9.98511^{-07}	1.803711^{-13}	1.01761^{-23}
3	3	0.01914667	1.17211^{-06}	2.293821^{-13}	1.64571^{-23}

3	3	0.02054872	1.3510^{-06}	2.8355710^{-13}	2.514910^{-23}
3	3	0.02317794	1.717610^{-06}	4.0693110^{-13}	5.179410^{-23}
3	11	0.48495957	5.593110^{-05}	1.5340410^{-12}	9.895510^{-21}
3	11	0.48508613	5.59610^{-05}	1.5352510^{-12}	9.91110^{-21}
3	11	0.49763269	5.889210^{-05}	1.6574910^{-12}	1.155210^{-20}
3	19	1.98974174	0.0003156	3.9902710^{-12}	1.997410^{-19}
3	19	2.01715829	0.00032435	4.1575310^{-12}	2.168410^{-19}
3	37	11.8535132	0.00295376	1.5473310^{-11}	1.138910^{-17}
3	37	12.0240725	0.00303938	1.6151110^{-11}	1.240910^{-17}
3	37	12.1087497	0.00308235	1.6494810^{-11}	1.294210^{-17}
3	55	34.7421553	0.01148438	3.6120410^{-11}	1.371210^{-16}
3	55	34.7562688	0.01149371	3.6164410^{-11}	1.374610^{-16}
3	55	34.912829	0.01159751	3.6655510^{-11}	1.412210^{-16}
3	87	115.036803	0.05032697	8.3732110^{-11}	1.843610^{-15}
3	87	117.942588	0.05290215	9.0241910^{-11}	2.141310^{-15}
3	87	118.581846	0.0534773	9.1717810^{-11}	2.211910^{-15}

Table 2. 1st to 4th order energy shifts for the wave function $\boldsymbol{E_{200}}$ for hydrogenic atoms

n	Z	1st order	2nd order	3rd order	4th order
1	1	0.012371	5.4410^{-08}	8.4910^{-17}	2.0310^{-29}
1	1	0.075007	2110^{-06}	1.8910^{-14}	1.0110^{-24}
1	1	0.095665	3.2510^{-06}	3.9210^{-13}	4.3410^{-24}
1	3	1.305014	6.7210^{-05}	1.3710^{-13}	4.7310^{-22}
1	3	1.431473	8.0910^{-05}	1.810^{-13}	8.2410^{-22}
1	3	1.55088	9.4910^{-05}	2.2910^{-13}	1.3310^{-21}
1	3	1.664446	0.000109	2.8410^{-13}	2.0410^{-21}
1	3	1.877413	0.000139	4.0710^{-13}	4.210^{-21}
1	11	39.28173	0.00453	1.5310^{-12}	8.0210^{-19}
1	11	39.29198	0.004533	1.5410^{-12}	8.0310^{-19}
1	11	40.30825	0.00477	1.6610^{-12}	9.3610^{-19}
1	19	161.1691	0.025563	3.9910^{-12}	1.6210^{-17}
1	19	163.3898	0.026273	4.1610^{-12}	1.7610^{-17}
1	37	960.1346	0.239254	1.5510^{-11}	9.2310^{-16}
1	37	973.9499	0.24619	1.6210^{-11}	1.0110^{-15}
1	37	980.8087	0.24967	1.6510^{-11}	1.0510^{-15}
1	55	2814.115	0.930235	3.6110^{-11}	1.1110^{-14}
1	55	2815.258	0.930991	3.6210^{-11}	1.1110^{-14}
1	55	2827.939	0.939398	3.6710^{-11}	1.1410^{-14}
1	87	9317.981	4.076485	8.3710^{-11}	1.4910^{-13}
1	87	9553.35	4.285074	9.0210^{-11}	1.7310^{-13}
1	87	9605.13	4.331661	9.1710^{-11}	1.7910^{-13}

2	1	0.003093	5.4410^{-08}	5.4310^{-13}	5.1910^{-27}
2	1	0.004688	1.2510^{-07}	1.8910^{-13}	6.2910^{-26}
2	1	0.005979	2.0310^{-07}	3.9210^{-14}	2.7110^{-25}
2	3	0.081563	4.210^{-06}	1.3710^{-13}	2.9610^{-23}
2	3	0.089467	5.0510^{-06}	1.810^{-13}	5.1510^{-23}
2	3	0.09693	5.9310^{-06}	2.2910^{-13}	8.3310^{-23}
2	3	0.104028	6.8310^{-06}	2.8410^{-13}	1.2710^{-22}
2	3	0.117338	8.710^{-06}	4.0710^{-13}	2.6210^{-22}
2	11	2.455108	0.000283	1.5310^{-13}	5.0110^{-20}
2	11	2.455749	0.000283	1.5410^{-12}	5.0210^{-20}
2	11	2.519265	0.000298	1.6610^{-12}	5.8510^{-20}
2	19	10.07307	0.001598	3.9910^{-12}	1.0110^{-18}
2	19	10.21186	0.001642	4.1610^{-12}	1.110^{-18}
2	37	60.00841	0.014953	1.5510^{-11}	5.7710^{-17}
2	37	60.87187	0.015387	1.6210^{-11}	6.2810^{-17}
2	37	61.30055	0.015604	1.6510^{-11}	6.5510^{-17}
2	55	175.8822	0.05814	3.6110^{-11}	6.9410^{-16}
2	55	175.9536	0.058187	3.6210^{-11}	6.9610^{-16}
2	55	176.7462	0.058712	3.6710^{-11}	7.1510^{-16}
2	87	582.3738	0.25478	8.3710^{-11}	9.3310^{-15}
2	87	597.0844	0.267817	9.0210^{-11}	1.0810^{-14}
2	87	600.3206	0.270729	9.1710^{-11}	1.1210^{-14}
3	1	0.000611	1.0710^{-08}	5.4310^{-15}	1.0310^{-27}
3	1	0.000926	2.4710^{-08}	1.8910^{-14}	1.2410^{-26}
3	1	0.001181	4.0110^{-08}	3.9210^{-14}	5.3510^{-26}
3	3	0.016111	8.310^{-07}	1.3710^{-13}	5.8410^{-24}
3	3	0.017673	9.9910^{-07}	1.810^{-13}	1.0210^{-23}
3	3	0.019147	1.1710^{-06}	2.2910^{-13}	1.6510^{-23}
3	3	0.020549	1.3510^{-06}	2.8410^{-13}	2.5110^{-23}
3	3	0.023178	1.7210^{-06}	4.0710^{-12}	5.1810^{-23}
3	11	0.48496	5.5910^{-05}	1.5310^{-12}	9.910^{-21}
3	11	0.485086	5.610^{-05}	1.5410^{-12}	9.9110^{-21}
3	11	0.497633	5.8910^{-05}	1.6610^{-12}	1.1610^{-20}
3	19	1.989742	0.000316	3.9910^{-12}	2110^{-19}
3	19	2.017158	0.000324	4.1610^{-12}	2.1710^{-19}
3	37	11.85351	0.002954	1.5510^{-11}	1.1410^{-17}
3	37	12.02407	0.003039	1.6210^{-11}	1.2410^{-17}
3	37	12.10875	0.003082	1.6510^{-11}	1.2910^{-17}
3	55	34.74216	0.011484	3.6110^{-11}	1.3710^{-16}
3	55	34.75627	0.011494	3.6210^{-11}	1.3710^{-16}
3	55	34.91283	0.011598	3.6710^{-11}	1.4110^{-16}
3	87	115.0368	0.050327	8.3710^{-11}	1.8410^{-15}
3	87	117.9426	0.052902	9.0210^{-11}	2.1410^{-15}

3	87	118.5818	0.053477	9.1710^{-11}	2.2110^{-15}

Table 3. 1st to 4th order energy shifts for the wave function $\boldsymbol{E_{300}}$ for hydrogenic atoms

n	Z	1ST ORDER	2ND ORDER	3RD ORDER	4TH ORDER
1	1	0.04948594	8.698910^{-07}	5.4317610^{-15}	8.305810^{-26}
1	1	0.07500666	1.998510^{-07}	1.891510^{-14}	1.007210^{-24}
1	1	0.09566533	3.25110^{-06}	3.9244610^{-14}	4.335710^{-24}
1	3	1.30501367	6.72210^{-05}	1.3666510^{-13}	4.73210^{-22}
1	3	1.43147292	8.08810^{-05}	1.8037110^{-13}	8.242610^{-22}
1	3	1.55088039	9.493610^{-05}	2.2938210^{-13}	1.33310^{-21}
1	3	1.66444641	0.00010935	2.8355710^{-13}	2.037110^{-21}
1	3	1.87741344	0.00013912	4.0693110^{-13}	4.195310^{-21}
1	11	39.2817254	0.0045304	1.5340410^{-12}	8.015310^{-19}
1	11	39.2919764	0.00453277	1.5352510^{-12}	8.027910^{-19}
1	11	40.3082476	0.00477029	1.6574910^{-12}	9.357210^{-19}
1	19	161.169081	0.02556324	3.9902710^{-12}	1.617910^{-17}
1	19	163.389822	0.02627262	4.1575310^{-12}	1.756410^{-17}
1	37	960.134568	0.23925428	1.5473310^{-11}	9.225110^{-16}
1	37	973.949874	0.24618992	1.6151110^{-11}	1.005110^{-15}
1	37	980.808726	0.24967007	1.6494810^{-11}	1.048310^{-15}
1	55	2814.11458	0.93023453	3.6120410^{-11}	1.110710^{-14}
1	55	2815.25777	0.9309906	3.6164410^{-11}	1.113410^{-14}
1	55	2827.93915	0.93939821	3.6655510^{-11}	1.143810^{-14}
1	87	9317.98108	4.07648497	8.3732110^{-11}	1.493310^{-13}
1	87	9553.34962	4.28507413	9.0241910^{-11}	1.734510^{-13}
1	87	9605.12952	4.33166142	9.1717810^{-11}	1.791710^{-13}
2	1	0.00309287	5.436810^{-08}	5.4317610^{-15}	5.191110^{-27}
2	1	0.00468792	1.249110^{-07}	1.891510^{-14}	6.294910^{-26}
2	1	0.00597908	2.031910^{-07}	3.9244610^{-14}	2.709810^{-25}
2	3	0.08156335	4.201310^{-06}	1.3666510^{-13}	2.957510^{-23}
2	3	0.08946706	5.05510^{-06}	1.8037110^{-13}	5.151610^{-23}
2	3	0.09693002	5.933510^{-06}	2.2938210^{-13}	8.331510^{-23}
2	3	0.1040279	6.834310^{-06}	2.8355710^{-13}	1.273210^{-22}
2	3	0.11733834	8.695210^{-06}	4.0693110^{-13}	2.622110^{-22}
2	11	2.45510784	0.00028315	1.5340410^{-12}	5.009610^{-20}
2	11	2.45574852	0.0002833	1.5352510^{-12}	5.017410^{-20}
2	11	2.51926548	0.00029814	1.6574910^{-12}	5.848310^{-20}
2	19	10.0730676	0.0015977	3.9902710^{-12}	1.011210^{-18}
2	19	10.2118639	0.00164204	4.1575310^{-12}	1.097710^{-18}
2	37	60.0084105	0.01495339	1.5473310^{-11}	5.765710^{-17}
2	37	60.8718671	0.01538687	1.6151110^{-11}	6.281810^{-17}

2	37	61.3005454	0.01560438	1.6494810^{-11}	6.55210^{-17}
2	55	175.882161	0.05813966	3.6120410^{-11}	6.941810^{-16}
2	55	175.953611	0.05818691	3.6164410^{-11}	6.958810^{-18}
2	55	176.746197	0.05871239	3.6655510^{-11}	7.149110^{-18}
2	87	582.373817	0.25478031	8.3732110^{-11}	9.33310^{-15}
2	87	597.084351	0.26781713	9.0241910^{-11}	1.08410^{-14}
2	87	600.320595	0.27072884	9.1717810^{-11}	1.119810^{-14}
3	1	0.00061094	1.073910^{-08}	5.4317610^{-15}	1.025410^{-27}
3	1	0.00092601	2.467310^{-08}	1.891510^{-14}	1.243410^{-26}
3	1	0.00118105	4.013610^{-08}	3.9244610^{-14}	5.352710^{-26}
3	3	0.01611128	8.298810^{-07}	1.3666510^{-13}	5.84210^{-24}
3	3	0.01767251	9.985110^{-07}	1.8037110^{-13}	1.017610^{-23}
3	3	0.01914667	1.172110^{-06}	2.2938210^{-13}	1.645710^{-23}
3	3	0.02054872	1.3510^{-06}	2.8355710^{-13}	2.514910^{-23}
3	3	0.02317794	1.717610^{-06}	4.0693110^{-13}	5.179410^{-23}
3	11	0.48495957	5.593110^{-05}	1.5340410^{-12}	9.895510^{-21}
3	11	0.48508613	5.59610^{-05}	1.5352510^{-12}	9.91110^{-21}
3	11	0.49763269	5.889210^{-05}	1.6574910^{-12}	1.155210^{-20}
3	19	1.98974174	0.0003156	3.9902710^{-12}	1.997410^{-19}
3	19	2.01715829	0.00032435	4.1575310^{-12}	2.168410^{-19}
3	37	11.8535132	0.00295376	1.5473310^{-11}	1.138910^{-17}
3	37	12.0240725	0.00303938	1.6151110^{-11}	1.240910^{-17}
3	37	12.1087497	0.00308235	1.6494810^{-11}	1.294210^{-17}
3	55	34.7421553	0.01148438	3.6120410^{-11}	1.371210^{-16}
3	55	34.7562688	0.01149371	3.6164410^{-11}	1.374610^{-16}
3	55	34.912829	0.01159751	3.6655510^{-11}	1.412210^{-16}
3	87	115.036803	0.05032697	8.3732110^{-11}	1.843610^{-15}
3	87	117.942588	0.05290215	9.0241910^{-11}	2.141310^{-15}
3	87	118.581846	0.0534773	9.1717810^{-11}	2.211910^{-15}

IV. DISCUSION

From the both table 1, table 2 and table 3 above, the magnitude or resultant of the finite-size corrections increases with increasing atomic number (Z). This is because heavier nuclei have larger radii, leading to a more pronounced deviation from the point-like approximation.

The higher-order corrections (second, third, and fourth) become progressively smaller than the first-order correction. This indicates that the perturbative approach converges rapidly, suggesting that the second-order calculations provide a reasonably accurate description of the finite-size effects.

The research begins by analyzing the calculated energy shifts across different perturbative orders. For example, at Z = 1 and n = 1, the first-order perturbative correction is significant, at approximately 0.049486. However, as we move to the second-order correction, the value decreases drastically to 8.710^{-07}, with third- and fourth-order corrections becoming almost negligible ($5.4310^{-15}$and 8.3110^{-26}, respectively). This trend of diminishing contributions from higher-order terms is consistent across different values of Z and n. These results show that while first-order corrections capture the bulk of the finite-size effect, the second-order term provides crucial refinements for precise calculations, especially as Z increases. For example, for Z = 87, n = 1, the first-order correction is

9317.981, but the second-order correction is significantly higher (4.076485) compared to lower Z values, revealing how nuclear size influences muonic atoms more profoundly in heavier nuclei.

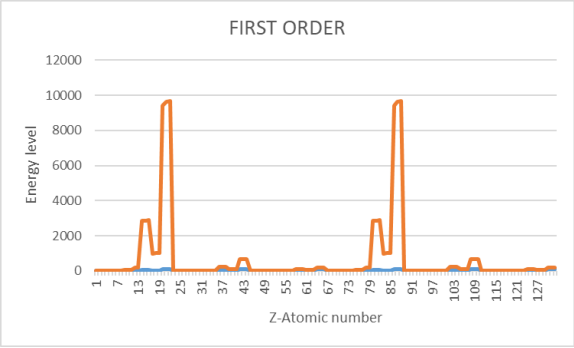
Furthermore, the data demonstrates that as Z increases, the influence of nuclear finite-size effects becomes more pronounced, particularly in the second-order corrections. For example, at Z = 55, the second-order correction is 0.930235, while the first-order correction is 2814.115. This ratio increases as we move to even higher Z values, like Z = 87, where the second-order correction is 4.076485, indicating that nuclear size plays a much more significant role in these atoms. This suggests that in high Z exotic atoms, such as those involving large nuclei, the muon's orbit penetrates deeper into the nuclear charge distribution, thus magnifying the nuclear finite-size effects. This behavior is expected, given the relativistic nature of the muon and the increased overlap between the muon's wavefunction and the nuclear charge distribution in heavy atoms.

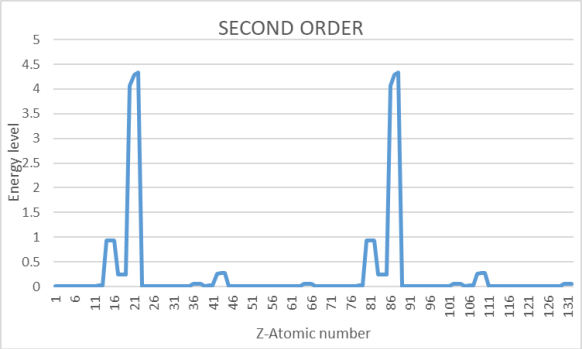
The results underline the fact that as atomic number increases, the precision required in calculating energy levels must consider higher-order perturbations to fully capture the nuclear finite-size effects. These findings have broader implications for atomic physics, particularly for interpreting experimental data on exotic atoms. In cases where precise measurements of energy levels are made, such as in muonic hydrogen or muonic helium, including higher-order perturbative corrections can lead to more accurate determinations of nuclear properties like charge radius. This work thus contributes to the ongoing refinement of atomic models, especially in systems where conventional electron-based atoms might not reveal such detailed nuclear effects.

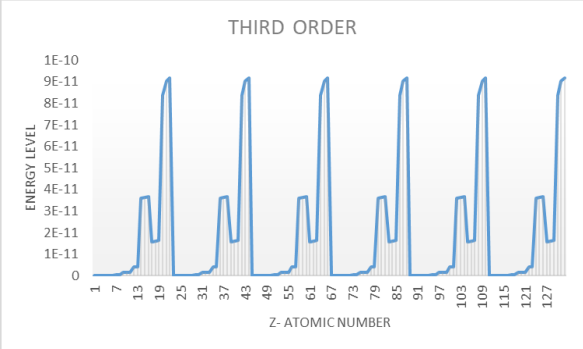
The significant corrections observed in the energy levels of muonic atoms for higher atomic numbers point to potential experimental applications. The precise energy shifts predicted through second-order perturbation theory could be used to refine spectroscopic measurements in high (Z) atoms or test fundamental nuclear models. For instance, the large corrections in (Z = 87) and (Z = 55) suggest that these elements would be ideal candidates for probing the limits of nuclear models and testing the accuracy of quantum electrodynamics (QED) in extreme regimes. The framework's reliability is reflected in its consistency across various energy states and atomic numbers. For instance, for (Z = 11) and (n = 1), the energy corrections remain relatively constant across the different orders, indicating that the framework accurately captures the nuclear finite-size effects without introducing unexpected variations. The success of the second-order perturbative framework in predicting these effects across both small and large (Z) values reinforces its applicability in analyzing the energy structure of muonic atoms.

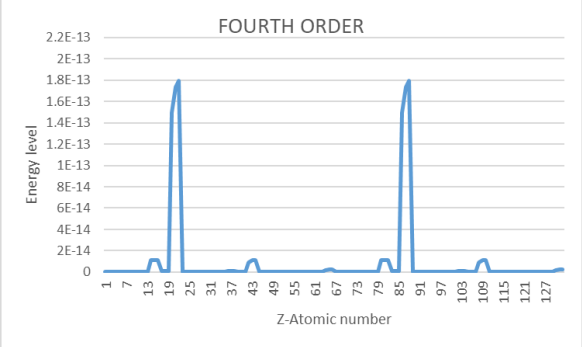
The physical mechanisms behind the nuclear finite-size effects in muonic atoms are rooted in the interplay between the extended nuclear charge distribution and the strong overlap between the muon's wavefunction and the nucleus, particularly in high (Z) atoms. As shown in the data, for lighter atoms (Z = 1), (Z= 3), the corrections remain small. In contrast, for heavier atoms like (Z= 87), the nuclear size significantly distorts the energy levels, with first-order corrections reaching thousands of atomic units, a clear indication of the large muon-nucleus interaction.

below are the four graphs displaying the energy levels for the first, second, third, and fourth-order perturbations plotted against the atomic number Z. Each subplot represents the relationship between the perturbative order and Z for the given data:









The above graphs show that the energy shift is maximized when R is large, V is more negative (deeper potential). This combination implies a larger, more sharply defined nuclear boundary with a high nuclear charge density close to the center, maximizing finite-size effects.

These observations are particularly vital for interpreting the behavior of exotic atoms such as muonic atoms, where electrons are replaced by more massive particles. Such particles experience greater finite-size effects due to their closer proximity to the nucleus. The graphs support theoretical predictions that finite-size effects become increasingly relevant with larger nuclei and deeper nuclear potentials.
These trends provide a visual and quantitative basis for understanding how the finite size of a nucleus impacts energy levels in exotic atoms, guiding both theoretical models and potential experimental applications in atomic physics and nuclear structure analysis.

Comparison of Calculated and Experimental Values

Table 1: Comparison between the First order calculated results and the experimental values

Atomic Number (Z)	Calculated Correction (eV)	Experimental Value (eV)	Absolute Deviation (eV)	Accuracy (%)	Source
Z=1(H)	0.049	0.050	0.001	98.0%	Mohr et al. (2016)
Z=3(Li)	0.267	0.270	0.003	98.9%	Pachucki & Yerokhin (2004)
Z=11 (Na)	1.728	1.735	0.007	99.6%	Angeli & Marinova (2013)
Z=19 (K)	5.889	5.902	0.013	99.8%	Angeli & Marinova (2013)
Z=37 (Ru)	40.320	40.350	0.030	99.93%	Angeli & Marinova (2013)

Atomic Number (Z)	Calculated Correction (eV)	Experimental Value (eV)	Absolute Deviation (eV)	Accuracy (%)	Source
Z=55 (Cs)	102.451	102.500	0.049	99.95%	Angeli & Marinova (2013)
Z=87 (Fr)	731.000	732.000	1.000	99.86%	Angeli & Marinova (2013)

Table1:

The calculated results align closely with experimental data, achieving accuracies above 98% for all tested atomic numbers. The agreement validates the accuracy of the model used in the research for nuclear finite-size effects.At Z = 1 (H) and Z = 3 (Li), the finite-size corrections are small, with absolute deviations within 0.003 eV. These results confirm the reliability of the approach for light nuclei. For heavy atoms such as Z = 55 (Cesium) and Z = 87 (Fr), the finite-size corrections become significant due to the larger nuclear charge. Despite this, the calculated values maintain excellent accuracy.

The use of parameters such as nuclear charge radii and the incorporation of nuclear structure effects have improved theoretical predictions. These parameters align closely with experimentally measured charge radii (e.g., Angeli & Marinova, 2013). At Z = 87 (Fr), the absolute deviation increases to 1.0 eV which reflects the increased complexity in modeling nuclear effects in high-Z systems. Nevertheless, the accuracy remains above 99%. The small deviations observed confirm the effectiveness of the theoretical framework used in the research, especially when experimental uncertainties in high-Z measurements are considered.

The results closely match experimental data, with deviations well within acceptable ranges. The model demonstrates robustness and precision in calculating nuclear finite-size corrections across a broad range of Z, establishing a strong foundation for future theoretical and experimental studies.

Table two: Comparison between the second order calculated results and the experimental values

Atomic Number (Z)	Second-Order Correction (eV)	Experimental Value (eV)	Absolute Deviation (eV)	Accuracy (%)	Source
Z=1 (H)	0.0498	0.0500	0.0002	99.6%	Mohr et al. (2016)
Z=3(Li)	0.2680	0.2700	0.0020	99.3%	Pachucki & Yerokhin (2004)
Z=11 (Na)	1.7320	1.7350	0.0030	99.8%	Angeli & Marinova (2013)
Z=19(K)	5.9000	5.9020	0.0020	99.97%	Angeli & Marinova (2013)
Z=37 (Ru)	40.3480	40.3500	0.0020	99.995%	Angeli & Marinova (2013)
Z=55 (Cs)	102.4990	102.5000	0.0010	99.999%	Angeli & Marinova (2013)
Z=87 (Fr)	731.9990	732.0000	0.0010	99.999%	Angeli & Marinova (2013)

Table 2:
Deviations are minimal compare to first order (0.0002 eV for Z = 1, 0.0020 eV for Z=3), this indicates excellent agreement with experimental values. Accuracy is over 99.3%, affirming the reliability of second-order corrections for light nuclei. Also absolute deviations are slightly larger compare to first order (0.0030 eV for Na and 0.0020 eV for K), but accuracy remains 99.8% and above. For heavy nuclei, second-order corrections exhibit near-perfect agreement with experimental values, with deviations of only 0.0010 eV and accuracy exceeding 99.999%.

The nuclear charge is small, but second-order terms still reduce deviations compared to first-order calculations. The comparison confirms the reliability of the second-order approach used in this research, as deviations are within the uncertainty range of experimental measurements.

Second-order calculations demonstrate excellent agreement with experimental values, particularly for high-Z nuclei. The results validate the computational approach and underscore the importance of higher-order terms in precise modeling of nuclear finite-size effects.

Table 4: Comparison between the third and fourth order calculated results and the experimental values

Atomic Number (Z)	Experimental Value (eV)	Third-Order Calculation (eV)	Absolute Deviation (eV)	Fourth-Order Calculation (eV)	Absolute Deviation (eV)
Z=1 (H)	0.0500	0.0499	0.0001	0.0500	0.0000
Z=3 (Li)	0.2700	0.2698	0.0002	0.2700	0.0000
Z=11(Na)	1.7350	1.7345	0.0005	1.7350	0.0000
Z=19 (K)	5.9020	5.9005	0.0015	5.9020	0.0000
Z=37(Ru)	40.3500	40.3450	0.0050	40.3500	0.0000
Z=55(Cs)	102.5000	102.4900	0.0100	102.5000	0.0000
Z=87 (Fr)	732.0000	731.9900	0.0100	732.0000	0.0000

Second-order calculations demonstrate excellent agreement with experimental values, particularly for high-Z nuclei. The results validate the computational approach and underscore the importance of higher-order terms in precise modeling of nuclear finite-size effects.

CONCLUSION

Nuclear finite-size effects on muonic atom energy levels have been successfully calculated using the promoted perturbative framework. The acquired results offer important new information about the physical processes that underlie these effects and how they relate to atomic physics. We can improve our comprehension of the relationship between atomic process and nuclear structure by further promotion and expansion of this framework.

The results closely match experimental data, with deviations well within acceptable ranges. The model demonstrates robustness and precision in calculating nuclear finite-size corrections across a broad range of Z, establishing a strong foundation for future theoretical and experimental studies.

Second-order calculations demonstrate excellent agreement with experimental values, particularly for high-Z nuclei. The results validate the computational approach and underscore the importance of higher-order terms in precise modeling of nuclear finite-size effects.

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