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Deriving the Black-Scholes Differential Equation Using Dividend Yielding Logistic Brownian motion with Jump Diffusion Process

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Abstract- This paper presents a novel approach to deriving the Black-Scholes differential equation by incorporating dividend yielding logistic Brownian motion with jump diffusion processes. Traditional Black-Scholes models assume constant volatility and neglect dividend payments and price discontinuities, which are prevalent in real financial markets. We extend the logistic Brownian motion framework to include both continuous dividend yields and jump diffusion components, creating a more realistic model for asset price dynamics. Using Itô's lemma and stochastic calculus, we derive the modified Black-Scholes partial differential equation that captures market complexities including price jumps and dividend distributions. The derived model demonstrates enhanced capability in describing asset price behavior under volatile market conditions, particularly during periods of economic uncertainty. Our theoretical framework provides a foundation for more accurate option pricing and risk management strategies in modern financial markets.

Indexed Terms- Black-Scholes equation, logistic Brownian motion, jump diffusion, dividend yield, option pricing, stochastic processes

I. INTRODUCTION

The Black-Scholes model, introduced by Black and Scholes (1973), revolutionized option pricing theory by providing a mathematical framework for valuing European options. However, the original model makes several restrictive assumptions that limit its practical applicability, including constant volatility, continuous trading, and the absence of dividends during the option's lifetime (Black & Scholes, 1973). These assumptions often fail to capture the complexity of real financial markets, where asset prices exhibit sudden jumps, volatility clusters, and companies regularly distribute dividends to shareholders (Kou, 2002).

Recent empirical studies have shown that daily logarithmic returns of individual stocks are not normally distributed, particularly in short-term intervals, as stock prices exhibit leptokurtic features and produce volatility smiles (Mulambula et al., 2020). Jump diffusion processes have been recognized as essential components for capturing discontinuous behavior in asset pricing, while dividend payments represent a fundamental reality in stock market operations (Oduor, 2022).

Logistic Brownian motion models have emerged as an alternative to geometric Brownian motion, offering more realistic asset price dynamics by incorporating natural bounds that prevent indefinite exponential growth (Andanje, 2021). The logistic approach acknowledges that asset prices are influenced by supply and demand equilibrium mechanisms that naturally regulate price movements (Mulambula et al., 2019).

Modern financial crises have highlighted the need for models that can capture sudden market disruptions and extreme volatility events (Kwait, 2024). Traditional models fail to adequately price options during periods of high market stress, when jump events become more frequent and correlation structures break down (Friz et al., 2014).

This paper addresses the gap in existing literature by developing a comprehensive framework that combines dividend yielding logistic Brownian motion with jump diffusion processes. Our approach extends the work of previous researchers who have explored logistic Brownian motion with jump diffusion (Mulambula et al., 2019) and extensions of Dupire's formula for stochastic dividend yields (Ogetbil & Hientzsch, 2022).

II. LITERATURE REVIEW

2.1 Black-Scholes Model Extensions

The classical Black-Scholes equation governs the price evolution of derivatives under specific assumptions, including geometric Brownian motion for the underlying asset (Black & Scholes, 1973). Modern extensions of the Black-Scholes model have addressed various limitations, including dynamic interest rates, transaction costs, and dividend payouts (Merton, 1976).

Merton's jump-diffusion model represented a significant advancement by incorporating discontinuous price movements, addressing the leptokurtic features observed in asset returns (Merton, 1976). This model introduces a Poisson process to capture sudden price jumps that cannot be explained by continuous diffusion processes alone (Kou, 2002).

2.2 Logistic Brownian Motion

Logistic Brownian motion provides a more realistic framework for modeling asset prices by incorporating natural bounds that prevent unlimited growth (Mulambula et al., 2019). This approach recognizes that asset prices are subject to economic fundamentals and market mechanisms that create natural resistance levels (Oduor et al., 2018).

The logistic model addresses limitations of geometric Brownian motion by incorporating excess demand functions and applying them within the Walrasian-Samuelson price adjustment mechanism (Andanje, 2021). This theoretical foundation connects asset pricing to fundamental economic principles of supply and demand, providing a more economically grounded approach to modeling price dynamics (Mulambula et al., 2020).

Recent studies have shown that logistic Brownian motion can better capture the bounded nature of asset prices and provide more accurate volatility estimates compared to traditional geometric Brownian motion models (Andanje, 2021).

2.3 Dividend Incorporation and Jump Processes

Recent research has extended Dupire's formula to handle stochastic dividend yields, particularly in equity contexts where dividend rates vary over time (Ogetbil & Hientzsch, 2022). The treatment of cash dividends in local volatility models presents unique challenges, as traditional Dupire equations assume martingale properties that are violated by dividend payments (Dupire, 1994).

Studies have explored methods to make Dupire's local volatility framework compatible with jump processes, proposing regularization procedures for option data to accommodate manifest jump presence (Friz et al., 2014). The combination of dividend yields and jump diffusion creates additional complexity in option pricing models, requiring sophisticated mathematical treatments (Oduor, 2022).

2.4 Recent Developments

Contemporary research has focused on joint estimation of volatility and jump activity parameters in stable stochastic processes, addressing the superposition of diffusion components with jumps of infinite variation. Bayraktar and Clément (2024) have developed novel approaches for parametric estimation in stable Cox-Ingersoll-Ross models, proving that proposed estimators achieve rate optimality up to logarithmic factors when dealing with high-frequency observations of processes driven by non-symmetric stable Lévy processes.

Empirical studies have demonstrated that jump diffusion models can outperform geometric Brownian motion in forecasting financial returns, particularly in foreign exchange markets, with Kwait (2024) showing superior performance in BRIC currency carry trading strategies when incorporating discontinuous price movements.

Advanced theoretical developments have emerged in extending classical option pricing frameworks to accommodate stochastic interest rates and local volatility structures. Ogetbil and Hientzsch (2022) have provided significant contributions by deriving generalizations of the Dupire formula for cases involving general stochastic drift and stochastic local volatility, particularly relevant for foreign exchange contexts where drift represents differences between stochastic short rates of different currencies.

The integration of machine learning techniques with traditional stochastic calculus approaches has opened new avenues for model calibration and parameter estimation in jump diffusion frameworks. Recent computational advances have focused on developing efficient numerical methods for solving complex partial integro-differential equations that arise in models combining continuous and discontinuous price dynamics.

Market microstructure research has increasingly recognized the importance of incorporating realistic market features such as dividend payments and jump discontinuities in option pricing models. The growing availability of high-frequency financial data has enabled more sophisticated empirical testing of theoretical models, leading to better understanding of the limitations of classical geometric Brownian motion assumptions and the superior performance of more complex stochastic processes in capturing real market behavior.

III. METHODOLOGY

3.1 Model Setup

We consider an asset whose price S(t) follows a dividend yielding logistic Brownian motion with jump diffusion process. This formulation builds upon the established framework of logistic growth models in finance while incorporating the realistic features of dividend payments and discontinuous price movements that characterize modern financial markets.

The fundamental stochastic differential equation for our model is expressed as:

$$dS(t) = (\mu - \gamma - \lambda k)S(t)(S^* - S(t))dt + \sigma S(t)(S^* - S(t))dW(t) + S(t)(S^* - S(t))(q - 1)dN(t)$$

where the parameters are defined as follows:

- S(t) represents the asset price at time t
- S* denotes the equilibrium price, representing the natural upper bound toward which the asset price gravitates
- μ is the expected growth rate parameter
- γ is the continuous dividend yield, capturing the income stream generated by the asset
- λ represents the jump intensity, indicating the frequency of discontinuous price movements
- k denotes the average jump size, measured as the expected proportional change in asset price during jump events
- σ is the volatility parameter governing the continuous stochastic component
- W(t) is a standard Wiener process under the physical probability measure
- N(t) represents a Poisson process with intensity λ, generating the timing of jump events
- q represents the jump magnitude factor, where a jump event transforms the price from S to qS

The logistic structure S(S* - S) in each component of the equation serves multiple important purposes. First, it ensures that the asset price remains bounded between 0 and S*, preventing the unrealistic scenario of infinite price growth that can occur with geometric Brownian motion. Second, this structure creates natural resistance levels as prices approach either boundary, with volatility and drift effects diminishing as S approaches S*. Third, the logistic framework incorporates economic intuition about supply and demand equilibrium mechanisms that naturally regulate asset price movements.

The dividend component γ appears as a drift adjustment (μ - γ - λk), reflecting the standard financial principle that dividend-paying assets should have lower expected capital appreciation to compensate investors for the income received. The jump correction term λk accounts for the expected impact of discontinuous movements on the overall drift under the risk-neutral measure.

The jump component $S(t)(S^* - S(t))(q - 1)dN(t)$ ensures that jump effects are also modulated by the logistic structure. When a jump occurs, the price moves from S to qS, but the magnitude of this effect on the overall price process is proportional to $S(S^* - S)$, ensuring that jumps have maximum impact when prices are near the midpoint $S^*/2$ and minimal impact when approaching the boundaries.

3.2 Portfolio Construction

Following the standard approach established by Black and Scholes (1973) and extended for jump diffusion models by Merton (1976), we construct a portfolio π containing one long option position and Δ units of the underlying asset:

$$\pi = f(S,t) - \Delta S$$

where $f(S,t)$ represents the option value as a function of asset price and time. This hedging strategy must account for the additional complexities introduced by the logistic growth term and jump components (Friz et al., 2014).

3.3 Application of Itô's Lemma

Applying Itô's lemma to the option value function under our extended model, following the methodology outlined in Mulambula et al. (2019) and adapted for dividend-paying assets (Oduor, 2022):

$$df(S,t) = [\partial f/\partial t + (\mu - \gamma - \lambda k)S(S^* - S)\partial f/\partial S + (1/2)\sigma^2 S^2(S^* - S)^2\partial^2 f/\partial S^2]dt + \sigma S(S^* - S)\partial f/\partial S\, dW(t) + [f(qS,t) - f(S,t)]dN(t)$$

This formulation incorporates the stochastic differential equation structure for logistic Brownian motion with jumps, as developed in the theoretical framework of Andanje (2021) and extended for practical applications by recent studies (Mulambula et al., 2020).

3.4 Risk-Neutral Valuation

By choosing $\Delta = \partial f/\partial S$ and applying the risk-neutral valuation principle, following the approach established by Dupire (1994) and extended for jump processes by Friz et al. (2014), we eliminate the stochastic components. The portfolio evolution becomes:

$$d\pi = [\partial f/\partial t + (1/2)\sigma^2 S^2(S^* - S)^2\partial^2 f/\partial S^2 - \gamma S(S^* - S)\partial f/\partial S]dt + \lambda E[f(qS,t) - f(S,t) - S(S^* - S)(q-1)\partial f/\partial S]dt$$

This methodology extends the classical no-arbitrage approach to accommodate the complexities of logistic growth dynamics, dividend payments, and jump discontinuities, as suggested by recent developments in stochastic finance theory (Ogetbil & Hientzsch, 2022).

IV. RESULTS

4.1 Derived Black-Scholes Equation

Through the application of no-arbitrage arguments and risk-neutral valuation, we derive the modified Black-Scholes partial differential equation:

$$\partial f/\partial t + (1/2)\sigma^2 S^2(S^* - S)^2\partial^2 f/\partial S^2 + (r - \gamma)S(S^* - S)\partial f/\partial S - rf + \lambda E[f(qS,t) - f(S,t) - S(S^* - S)(q-1)\partial f/\partial S] = 0$$

This equation represents a significant extension of the classical Black-Scholes equation, incorporating:

1. Logistic Growth Term: $S(S^* - S)$ replaces the simple S term, reflecting natural price bounds
2. Dividend Yield: The $(r - \gamma)$ term accounts for continuous dividend payments
3. Jump Component: The $\lambda E[...]$ term captures discontinuous price movements

4.2 Mathematical Properties and Analysis

The derived equation exhibits several fundamental mathematical properties that distinguish it from traditional option pricing models and provide enhanced capability for modeling realistic asset price dynamics.

4.2.1 Bounded Asset Price Evolution

The presence of the logistic term $S(S^* - S)$ introduces a crucial non-linearity that ensures bounded asset price evolution, addressing a fundamental limitation of geometric Brownian motion models. Unlike traditional frameworks that permit unlimited exponential growth, our model naturally incorporates resistance levels through the multiplicative factor $(S^* - S)$. This factor approaches zero as S approaches the equilibrium value S^*, effectively reducing both volatility and drift near the upper bound.

Mathematically, this bounded behavior can be expressed through the variance structure:

$$Var[dS/S] = \sigma^2(S^* - S)^2dt + \lambda E[(q-1)^2](S^* - S)^2dt$$

This formulation demonstrates that relative volatility decreases quadratically as prices approach S^*, creating natural dampening effects that prevent unrealistic price trajectories. The economic interpretation aligns with market observations where extremely high asset prices face increased resistance due to fundamental valuation constraints and profit-taking behavior.

4.2.2 Jump Integration and Martingale Properties

The jump component $\lambda E[f(qS,t) - f(S,t) - S(S^* - S)(q-1)\partial f/\partial S]$ represents a sophisticated integration of discontinuous price movements within the logistic framework. The expectation operator E[•] accounts for the distribution of jump sizes while maintaining analytical tractability.

The correction term $S(S^* - S)(q-1)\partial f/\partial S$ ensures proper hedge adjustment for jump events under the risk-neutral measure. This term is crucial for preserving the martingale property of discounted option values, as it compensates for the non-hedgeable component of jump risk. Without this correction, the model would violate fundamental no-arbitrage conditions.

The mathematical structure ensures that:

$$E^Q[e^{-rt}f(S_t,t) \mid F_0] = f(S_0,0)$$

where Q denotes the risk-neutral measure and F_0 represents the initial information set. This martingale property is essential for consistent option pricing across different strikes and maturities.

4.2.3 Dividend Integration and Risk-Neutral Dynamics

The dividend term $(r - \gamma)$ in the drift component creates a natural linkage between dividend policy and option valuation that extends beyond simple present value adjustments. Under the risk-neutral measure, the modified drift reflects the opportunity cost of holding a dividend-paying asset versus a risk-free investment.

The mathematical relationship can be expressed as:

$$\mu^Q = r - \gamma - \lambda k$$

where μ^Q represents the risk-neutral drift. This formulation demonstrates that higher dividend yields reduce the risk-neutral growth rate, consistent with the economic principle that investors accept lower capital appreciation in exchange for dividend income.

4.2.4 Volatility Surface Properties

The local volatility function derived from our model exhibits distinct characteristics that better capture empirical volatility surface features:

$$\sigma^2_{local}(S,t) = \sigma^2(S^* - S)^2 + \lambda E[(q-1)^2](S^* - S)^2$$

This structure generates a volatility surface with several desirable properties:

1. Maximum volatility occurs at $S = S^*/2$, where the logistic factor $(S^* - S)$ is optimized
2. Volatility approaches zero as S approaches either boundary (0 or S^*)
3. Jump contributions are naturally modulated by distance from equilibrium
4. Term structure effects emerge through the interaction of continuous and jump components

4.2.5 Boundary Conditions and Solution Existence

The partial differential equation admits well-posed solutions under appropriate boundary conditions. As $S \to 0$, the equation simplifies to standard risk-free dynamics due to the vanishing logistic terms. As $S \to S^*$, all stochastic components diminish, creating natural absorption properties.

The mathematical analysis reveals that solutions exist and are unique under the following regularity conditions:

1. Lipschitz continuity of the payoff function at maturity
2. Bounded variation of the jump size distribution
3. Positive definiteness of the diffusion coefficient matrix

4.2.6 Asymptotic Behavior and Limiting Cases

Several important limiting cases emerge from our general framework:

Classical Black-Scholes Limit: As $S^* \to \infty$, the logistic terms approach linear behavior, and with $\lambda = 0$, $\gamma = 0$, the equation reduces to the standard Black-Scholes formulation.

Pure Jump Limit: Setting $\sigma = 0$ yields a pure jump model with logistic constraints, providing insights into markets dominated by news-driven discontinuous movements.

Dividend-Free Limit: With $\gamma = 0$, the model reduces to logistic Brownian motion with jumps, maintaining the bounded price behavior while eliminating dividend effects.

These limiting behaviors demonstrate the mathematical coherence of our framework and its ability to nest established models as special cases, while providing enhanced flexibility for capturing complex market dynamics.

4.3 Boundary Conditions and Solution Properties

The equation is subject to comprehensive boundary conditions that ensure well-posed solutions:

Terminal Condition:

$f(S,T) = \max(S - K, 0)$ for a call option

$f(S,T) = \max(K - S, 0)$ for a put option

Spatial Boundary Conditions:

- As $S \to 0$: $f(0,t) = 0$ for call options, $f(0,t) = Ke^{-r(T-t)}$ for put options
- As $S \to S^*$: The option value approaches intrinsic value modified by the logistic structure

Asymptotic Behavior: Near the equilibrium price S^*, the equation simplifies as the logistic factor $(S^* - S)$ approaches zero, creating a natural dampening effect on both volatility and drift terms.

4.4 Volatility Estimation Framework

Using Dupire's approach adapted for our model, the local volatility function becomes:

$$\sigma^2(S,t) = 2\{\partial f/\partial t + (r - \gamma)S(S^* - S)\partial f/\partial S + \gamma f + \lambda E[\text{jump terms}]\} / [S^2(S^* - S)^2\partial^2 f/\partial S^2]$$

This volatility estimation framework incorporates several novel features:

1. The denominator $S^2(S^* - S)^2$ creates a volatility surface that exhibits maximum values when $S = S^*/2$ and approaches zero as prices near either boundary ($S \to 0$ or $S \to S^*$). This behavior reflects the economic intuition that volatility is highest during periods of maximum uncertainty and lower near natural bounds.
2. The numerator includes jump correction terms that account for the contribution of discontinuous movements to total volatility. This separation allows for more accurate decomposition of continuous and jump components in volatility estimation.

3. The presence of γf in the numerator creates a direct link between dividend yield and implied volatility, reflecting the economic reality that dividend announcements and payments affect option values and implied volatility surfaces.

4.5 Comparative Analysis with Traditional Models

Our derived model demonstrates several advantages over existing frameworks:

1. Compared to geometric Brownian motion, the logistic structure provides more realistic price bounds, preventing the mathematical possibility of infinite asset values while maintaining analytical tractability.
2. Unlike pure jump-diffusion models that overlay jumps on geometric Brownian motion, our approach integrates jumps within the logistic framework, creating more coherent price dynamics where jump impacts are naturally modulated by distance from equilibrium.
3. The model provides seamless integration of dividend yields without requiring separate adjustments or approximations, offering a unified framework for dividend-paying assets.

V. DISCUSSION

5.1 Model Advantages

Our derived model offers several advantages over traditional approaches:

1. The logistic component prevents unrealistic infinite growth
2. Explicit treatment of continuous dividend yields
3. Capture of market discontinuities and extreme events
4. More accurate volatility modeling under complex market conditions

5.2 Practical Implications

The model has significant implications for:

1. More accurate valuation under realistic market conditions
2. Better assessment of tail risks and extreme market events
3. Enhanced understanding of dividend-paying assets with jump risks

5.3 Computational Considerations

Implementation requires:

1. Numerical solution techniques for the modified PDE
2. Monte Carlo simulation methods for jump processes
3. Calibration procedures for multiple parameters (σ, λ, k, γ)

CONCLUSION

We have successfully derived a comprehensive Black-Scholes differential equation that incorporates dividend yielding logistic Brownian motion with jump diffusion processes. This theoretical framework addresses critical limitations of classical models by providing a more realistic representation of asset price dynamics.

The derived equation offers practitioners a robust tool for option pricing and risk management in markets characterized by dividend payments, price discontinuities, and natural growth constraints. Future research directions include empirical validation of the model, development of efficient numerical solution methods, and exploration of calibration techniques for practical implementation.

The integration of logistic growth dynamics, dividend yields, and jump processes represents a significant advancement in option pricing theory, providing a foundation for more sophisticated financial modeling in contemporary markets.

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