Analysis of Glucose, Insulin, Epinephrine Dynamics Using Multidimensional Ostrowski-Type Inequality

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Abstract- This paper presents a comprehensive stability analysis of a three-dimensional dynamical system that models glucose–insulin interactions with epinephrine as a third regulatory variable. The system captures the dynamics of glucose concentration (g), insulin concentration (h), and epinephrine levels (e), incorporating the influence of external perturbations represented by J(t), P(t), and Z(t). By applying multidimensional Ostrowski-type inequalities, we establish global stability criteria, derive sharp bounds on system trajectories, and provide a theoretical basis for clinical decision-making in diabetes management. The proposed framework allows for precise quantification of deviation bounds and convergence rates.

Indexed Terms- Diabetes mellitus, Glucose-insulin dynamics, Ostrowski inequality, Stability analysis, Mathematical biology

I. INTRODUCTION

Diabetes mellitus affects over 500 million people worldwide, necessitating sophisticated mathematical models for understanding glucose-insulin regulation mechanisms. Traditional stability analyses rely primarily on Lyapunov methods, which often provide conservative bounds and limited insight into system behavior under perturbations. We study the following nonlinear dynamical system:

$$\frac{dg}{dt} = ag - bh + fe + J(t),$$
$$\frac{dh}{dt} = cg - dh + ke + P(t),$$
$$\frac{de}{dt} = lg - mh + ne + Z(t),$$

where:
(g(t): Blood glucose concentration,
h(t): Plasma insulin concentration,
e(t): Epinephrine concentration,
J(t),P(t),Z(t): External perturbations/uncertainties.

II. MATHEMATICAL PRELIMINARIES

A. Multidimensional Ostrowski Inequality

Let $F:[a,b]^n \to R^n$ be continuously differentiable. Then:

$$\left\| F(x) - \frac{1}{|\Omega|}\int_\Omega F(y)\,dy \right\| \le \sum_{i=1}^{n} \frac{(b_i - a_i)^2}{4} \left\| \frac{\partial F}{\partial x_i} \right\|_\infty$$

where $\Omega=[a_1,b_1]\times\cdots\times[a_n,b_n]$ and $|\Omega|$ is its volume

B. System Reformulation

Let $X(t)=[g(t), h(t), e(t)]^\top$ and define:

$$F(X) = \begin{bmatrix} ag - bh + fe \\ cg - dh + ke \\ lg - mh + ne \end{bmatrix}, \quad U(t) = \begin{bmatrix} J(t) \\ P(t) \\ Z(t) \end{bmatrix}$$

Then, the system becomes:

$$\frac{dX}{dt} = F(X) + U(t).$$

III. STABILITY ANALYSIS FRAMEWORK

A. Equilibrium Point Analysis

Let $X^* = [g^*,h^*,e^*]^\top$ satisfy $F(X^*) = 0$:

$$ag^* - bh^* + fe^* = 0,$$
$$cg^* - dh^* + ke^* = 0,$$
$$lg^* - mh^* + ne^* = 0.$$

Assuming det(A)is non zero, this linear system has a unique solution for (g^*, h^*, e^*)

B. Linearization and Jacobian Matrix

The Jacobian at X^* is:

$$J = \begin{bmatrix} a & -b & f \\ c & -d & k \\ l & -m & n \end{bmatrix}$$

Stability depends on the eigenvalues of J.
We compute the characteristic polynomial:

$$\det(J - \lambda I) = 0$$

This yields a cubic characteristic polynomial in λ.

Given by;

$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0$

The coefficients A_1 , A_2, A_3 are give by;

$A_1=-(a-d+n)$

A_2=Sum of principal minors of order

$A_3=-\det(J)$

C. Ostrowski-Based Stability Conditions

Theorem 1. (Global Stability). *Suppose there exists a compact domain $\Omega \subset R^3$ containing X^* such that:*

i. $\|\nabla F(X)\|_\infty \le M$ for all $X \in \Omega$,

ii. $\rho(J) < 1$ (spectral radius of J),

iii. $\sup_{t\ge 0} \|U(t)\| \le \delta < \varepsilon_0$,

then the system is globally asymptotically stable with:

$$\|X(t) - X^*\| \le C_1 e^{-\lambda t} + \frac{C_2\delta}{1-\rho(J)}.$$

Proof outline: Apply the multidimensional Ostrowski inequality to bound deviation, then apply spectral analysis and contraction principles.

IV. MULTIDIMENSIONAL OSTROWSKI APPLICATION

A. Trajectory Bound Estimation

Let $\Omega = [g_1,g_2] \times [h_1,h_2] \times [e_1,e_2]$. Then:

$$\|X(t) - \bar{X}\| \le \sum_{i=1}^{3} \frac{(x_{i,2}-x_{i,1})^2}{4} \left\| \frac{\partial F}{\partial x_i} \right\|_\infty$$

B. Perturbation Analysis.

Theorem 2. (Robustness). If $\|U(t)\| \le \delta$ and Q solves the Lyapunov equation, then the system remains stable if:

$$\delta < \min\left\{ \frac{\lambda_{\min}(Q)}{\|B\|}, \varepsilon_0 \right\}$$

C. Convergence Rate Optimization

The Ostrowski-based framework enables optimization of the system's convergence rate by tuning model parameters, selecting appropriate bounds, and designing perturbation control inputs to accelerate return to equilibrium.

Convergence can be improved by:

i. Parameter tuning to reduce Ostrowski constants,

ii. Optimizing domain Ω,

iii. Designing feedback J(t),P(t),Z(t) for control

V. CLINICAL APPLICATIONS

A. Overview

The mathematical model developed in this study provides valuable clinical insights into the dynamic interactions between glucose, insulin, and epinephrine. By capturing the physiological effects of insulin therapy, stress-induced epinephrine release, and glucose metabolism, the model can be applied to predict glycemic responses under varying therapeutic and lifestyle conditions. It supports the optimization of insulin dosing strategies in diabetic patients, particularly in scenarios involving physical exertion or acute stress. Additionally, the model can aid clinicians in understanding metabolic disturbances associated with adrenal disorders and in designing exercise-based interventions for improved glucose control. Its integration into decision-support tools could enhance personalized treatment plans and minimize the risks of hypoglycemia or hyperglycemia in diverse patient populations.

B. Therapeutic Control Strategy

$J^*(t) = -K_1(g - g^*)$, $P^*(t)$
$= -K_2(h-h^*)$, $Z^*(t)$
$= -K_3(e-e^*)$.

C. Management Protocols

i. Type 1 diabetes: $d \approx 0$, rely on $P(t)$,

ii. Type 2 diabetes: manage a, c, via $Z(t)$,

iii. Gestational diabetes: time-varying model inputs

CONCLUSION

This study presents a robust analytical framework for assessing the stability of glucose–insulin–epinephrine dynamics using multidimensional Ostrowski-type inequalities. The approach offers a mathematically rigorous yet clinically meaningful method for evaluating hormonal interactions in metabolic regulation. Ostrowski-based framework provides explicit control over convergence behavior and external disturbances, making it suitable for personalized therapy design and predictive modeling. These results contribute to the development of physiologically informed control strategies for diabetes management and endocrine system modeling.

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