© JUN 2025 | IRE Journals | Volume 8 Issue 12 | ISSN: 2456-8880

Applications of α-Modified Sumudu Transform to Real-Life Problems

SAN LYNN AUNG[1], KHIN THAN SINT[2]
[1]*Department of Mathematics, Pathein University, Myanmar*
[2]*Department of Mathematics, University of Mandalay, Myanmar*

Abstract- Firstly, the definitions of α-modified Sumudu transform and its inverse are presented. Next, some properties of α-modified Sumudu transform are described. Moreover, the existence and uniqueness theorems for α-modified Sumudu transform are proved by using comparison test and α-modified Laplace and α-modified Sumudu duality. And then, α-modified Sumudu transforms of some elementary functions are discussed. Finally, α-modified Sumudu transform method is applied to find the solutions of real-life problems.

Indexed Terms- α-modified Sumudu transform, differential equations

I. INTRODUCTION

There are many integral transforms to solve differential equations and control engineering problems. They are Laplace, Fourier, Mellin and Hankel transforms. Among them, Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. A new integral transform, called the Sumudu transform defined for functions of exponential order, is proclaimed. The Sumudu transform is an integral transform similar to the Laplace transform, originally introduced by Watugala G. K. to solve the differential equations and control engineering problems in 1993. This transformation is defined over the set of the functions

$$H = \{g(t) \mid \exists M, \sigma_1, \sigma_2 > 0 : |g(t)| < Me^{\frac{|t|}{\sigma_j}}, \text{if } t \in (-1)^j \times [0, \infty)\}$$

by

$$S[g(t); w] = \frac{1}{w}\int_0^\infty g(t)e^{-\frac{t}{w}}\,dt = A(w),$$

$w \in (-\sigma_1, \sigma_2)$.

Moreover, Belgacem F. B. M. investigated further properties of this transformation in 2002. Some mathematicians were applied to this transformation to solve the differential equations [Eltayeb and Kilicman, 2010]. In this paper, we will consider the substitution the symbol 'α' instead of the exponential of the Sumudu transform. This new transform is called the α-modified Sumudu transform.

The rest of this paper is organized as follow: In Section 2, the basic concepts of α-modified Sumudu transform and its properties are presented. The main results are discussed in Section 3 by finding the solutions of real-life problems using α-modified Sumudu transform.

I. II. USEFUL DEFINITIONS AND PROPERTIES

In this section, the basic concepts of α-modified Sumudu transform are expressed by base on [Kilicman, Eltayed and Ismail, 2012] and its properties are examined by base on [Belgacem, Karaballi, Kalla, 2003].

2.1 Definition

A function g is said to be of *exponential order* $\frac{1}{\sigma}$, if there are positive constants $\frac{1}{\sigma}$ and M such that

$$|g(t)| \le M\alpha^{\frac{t}{\sigma}}, \ t \ge 0.$$

2.2 Definitions

The *α-modified Sumudu transform* defined for functions of exponential order is given over the set of the functions

$$H_\alpha = \{g(t) \mid \exists M, \sigma_1, \sigma_2 > 0 : |g(t)| < M\alpha^{\frac{|t|}{\sigma_j}},$$

$$\text{if } t \in (-1)^j \times [0, \infty)\} \quad (1)$$

by

$$S_\alpha[g(t); w] = \frac{1}{w}\int_0^\infty g(t)\alpha^{-\frac{t}{w}} dt = A_\alpha(w),$$

$$w \in (-\sigma_1, \sigma_2). \quad (2)$$

The function g(t) is called the *inverse α-modified Sumudu transform* or *inverse* of $A_\alpha(w)$ and will be denoted by $S_\alpha^{-1}[A_\alpha(w)]$, that is, we shall write

$$g(t) = S_\alpha^{-1}[A_\alpha(w)]. \quad (3)$$

2.3 Linear Property

If f and g are functions of t, and λ is a constant, then

(i) $S_\alpha[f(t) \pm g(t)] = S_\alpha[f(t)] \pm S_\alpha[g(t)]$,

(ii) $S_\alpha[\lambda g(t)] = \lambda S_\alpha[g(t)]$.

We can easily prove by using linear property of integral.

2.4 Change of Scale Property

If the α-modified Sumudu transform of g(t) is $A_\alpha(w)$ and k is a constant, then $S_\alpha[g(kt)] = A_\alpha(kw)$.

By definition, we have

$$S_\alpha[g(kt)] = \frac{1}{kw}\int_0^\infty g(u)\alpha^{-\frac{u}{kw}} du = A_\alpha(kw).$$

2.5 The α-Modified Sumudu Transform of Unit Step Function

The unit step function $U(t-a)$ is defined as follows:

$$U(t-a) = \begin{cases} 0, & t < a, \\ 1, & t > a, \end{cases}$$

where $a \ge 0$.

The α-modified Sumudu transform of unit step function is

$$S_\alpha[U(t-a)] = \frac{1}{w}\int_a^\infty \alpha^{-\frac{t}{w}} dt = \frac{\alpha^{-\frac{a}{w}}}{\log \alpha}.$$

2.6 First Shifting Theorem

Let the α-modified Sumudu transform of $g(t) \in H_\alpha$ be $A_\alpha(w)$. Then

$$S_\alpha[\alpha^{ct} g(t)] = \frac{1}{1-cw} A_\alpha\left(\frac{w}{1-cw}\right).$$

Proof:

By definition, we have

$$S_\alpha[\alpha^{ct} g(t)] = \frac{1}{w}\int_0^\infty g(t)\alpha^{-t\left(\frac{1-cw}{w}\right)} dt.$$

Let $v = t(1-cw)$, then $dv = (1-cw)dt$.

$$S_\alpha[\alpha^{ct} g(t)] = \frac{1}{w(1-cw)}\int_0^\infty g\left(\frac{v}{1-cw}\right)\alpha^{-\frac{v}{w}} dv$$

$$= \frac{1}{1-cw} A_\alpha\left(\frac{w}{1-cw}\right).$$

2.7 Second Shifting Theorem

If the α-modified Sumudu transform of g(t) is $A_\alpha(w)$, then

$$S_\alpha[g(t-a)H(t-a)] = \alpha^{-\frac{a}{w}} A_\alpha(w),$$

where $H(t-a)$ is the unit step function.

Proof:

By definition, we have

$$S_\alpha[g(t-a)H(t-a)] = \frac{1}{w}\int_a^\infty g(t-a)\alpha^{-\frac{t}{w}} dt$$

$$= \frac{1}{w}\int_0^\infty g(v)\alpha^{-\left(\frac{a+v}{w}\right)} dv$$

$$= \alpha^{-\frac{a}{w}} A_\alpha(w).$$

2.8 Existence Theorem of α-Modified Sumudu Transform

If g(t) is piecewise continuous on $[0, \infty)$ and of exponential order, then the α-modified Sumudu transform of g(t) exists for $|w| < \sigma$.

Proof:

We need to prove that the integral $\frac{1}{w}\int_0^\infty g(t)\alpha^{-\frac{t}{w}} dt$ or $\frac{1}{w}\int_0^\infty g(t)e^{-\frac{t\log\alpha}{w}} dt$ converges for $|w| < \sigma$. We begin by breaking up this integral into two separate integrals:

$$\frac{1}{w}\left[\int_0^R g(t)e^{-\frac{t\log\alpha}{w}}dt+\int_R^\infty g(t)e^{-\frac{t\log\alpha}{w}}dt\right], \qquad (4)$$

where R is chosen so that inequality $|g(t)| \le M\alpha^{\frac{t}{\sigma}}$ holds. As g(t) is piecewise continuous on $[0,\infty)$, so g(t) is piecewise continuous on $[0,R]$ of $[0,\infty)$.

Then $g(t)\exp\left(-\frac{t\log\alpha}{w}\right)$ is piecewise continuous on the interval $[0,R]$ and hence the first integral in (4) exists. Next, we have to prove the second integral in (4) converges by using comparison test. Thus, for all $t \ge R$,

$$\left|g(t)\exp\left(-\frac{t\log\alpha}{w}\right)\right| = |g(t)|\exp\left(-\frac{t\log\alpha}{w}\right)$$

$$\le M\exp\left(-t(\frac{\log\alpha}{w}-\frac{\log\alpha}{\sigma})\right)$$

Now, for $|w| < \sigma$,

$$\int_R^\infty M\exp\left(-t(\frac{\log\alpha}{w}-\frac{\log\alpha}{\sigma})\right)dt$$

$$=\frac{M\exp\left(-R(\frac{\log\alpha}{w}-\frac{\log\alpha}{\sigma})\right)}{\frac{\log\alpha}{w}-\frac{\log\alpha}{\sigma}} < \infty .$$

Thus, the improper integral of the larger function converges for $|w| < \sigma$, then by comparison test, the integral $\int_R^\infty \exp\left(-t(\frac{\log\alpha}{w}-\frac{\log\alpha}{\sigma})\right)dt$ converges for $|w| < \sigma$. Finally, the second integral in (4) exists. Therefore, the α-modified Sumudu transform of g(t) exists for $|w| < \sigma$.

2.9 The α-Modified Laplace and α-Modified Sumudu Duality

The α-modified Laplace transform of the function g(t) is given by

$$F_\alpha(s) = L_\alpha[g(t)] = \int_0^\infty g(t)\alpha^{-st}dt,\ \mathrm{Re}(s) > 0 .$$

The α-modified Laplace and α-modified Sumudu transforms exhibit a duality relation expressed as follows:

$$A_\alpha(w) = \frac{1}{w}F_\alpha\left(\frac{1}{w}\right) \text{ and } F_\alpha(s) = \frac{1}{s}A_\alpha\left(\frac{1}{s}\right).$$

Proof:

By definition, we have

$$A_\alpha(w) = \frac{1}{w}\int_0^\infty g(t)\alpha^{-\frac{t}{w}}dt = \frac{1}{w}F_\alpha\left(\frac{1}{w}\right)$$

and

$$F_\alpha(s)=\int_0^\infty g(t)\alpha^{-st}dt=\frac{1}{s}A_\alpha\left(\frac{1}{s}\right).$$

2.10 Uniqueness Theorem of α-Modified Sumudu Transform

Let $f(t)$ and $g(t)$ be continuous functions defined for $t \ge 0$ and have α-modified Sumudu transforms, $B_\alpha(w)$ and $A_\alpha(w)$, respectively. If $B_\alpha(w) = A_\alpha(w)$, then $f(t) = g(t)$, where w is a complex number.

Proof:

By using α-modified Laplace and α-modified Sumudu duality, we get

$$\frac{1}{w}G_\alpha\left(\frac{1}{w}\right) = \frac{1}{w}F_\alpha\left(\frac{1}{w}\right),$$

$$L_\alpha[f(t)] = L_\alpha[g(t)],$$

where the variable of these transforms is $\frac{1}{w}$.

By uniqueness theorem of α-modified Laplace transform, we obtain

$$f(t) = g(t) .$$

2.11 The α-Modified Sumudu Transforms of Function Derivatives

If the α-modified Sumudu transform of g(t) is $A_\alpha(w)$, then

(i) $S_\alpha[g'(t)] = \frac{1}{w}\left[(\log\alpha)A_\alpha(w) - g(0)\right]$, (5)

(ii) $S_\alpha[g''(t)] = \frac{1}{w^2}[(\log\alpha)^2A_\alpha(w) - (\log\alpha)g(0) - wg'(0)]$, (6)

(iii) $S_\alpha[g^{(n)}(t)] = \frac{1}{w^n}[(\log\alpha)^nA_\alpha(w) - \sum_{k=0}^{n-1}(\log\alpha)^{n-k-1}w^kg^{(k)}(0)]$, (7) where $g^{(0)} = g(0)$ and $g^{(k)}(0)$, $k = 1, 2, 3, ..., n-1$ is the k^{th} order derivative of the function g(t).

(i) Indeed, we have

$$S_\alpha[g'(t)] = \frac{1}{w}\int_0^\infty g'(t)\alpha^{-\frac{t}{w}}dt.$$

Using the integration by parts, we get

$$S_\alpha[g'(t)] = \frac{1}{w}\left[(\log\alpha)A_\alpha(w) - g(0)\right].$$

(ii) Indeed, let $f(t) = g'(t)$, we get

$$S_\alpha[f'(t)] = \frac{1}{w}(\log\alpha)S_\alpha[f(t)] - \frac{1}{w}f(0).$$

Then,

$$S_\alpha[g''(t)] = \frac{1}{w}(\log\alpha)S_\alpha[g'(t)] - \frac{1}{w}g'(0)$$

$$= \frac{1}{w^2}[(\log\alpha)^2 A_\alpha(w) - (\log\alpha)g(0) - wg'(0)].$$

We can easily prove by mathematical induction.

2.12 The α-Modified Sumudu Transforms of Some Elementary Functions

We consider the α-modified Sumudu transforms of some following elementary functions.

(i) If $g(t) = 1$, then

$$S_\alpha[g(t)] = S_\alpha[1] = \frac{1}{w}\int_0^\infty \alpha^{-\frac{t}{w}}dt = \frac{1}{\log\alpha}.$$

(ii) If $g(t) = t$, then

$$S_\alpha[g(t)] = S_\alpha[t] = \frac{1}{w}\int_0^\infty t\alpha^{-\frac{t}{w}}dt.$$

Using the integration by parts, we get

$$S_\alpha[t] = \frac{w}{(\log\alpha)^2}.$$

(iii) If $g(t) = t^2$, then

$$S_\alpha[g(t)] = S_\alpha[t^2] = \frac{1}{w}\int_0^\infty t^2\alpha^{-\frac{t}{w}}dt.$$

Using the integration by parts, we get

$$S_\alpha[t^2] = \frac{2}{\log\alpha}\int_0^\infty t\alpha^{-\frac{t}{w}}dt = \frac{2w^2}{(\log\alpha)^3}.$$

In the general case, if n is a positive integer, then

$$S_\alpha[t^n] = \frac{n!w^n}{(\log\alpha)^{n+1}}.$$

(iv) If $g(t) = e^{at}$, then

$$S_\alpha[g(t)] = S_\alpha[e^{at}] = \frac{1}{w}\int_0^\infty e^{-t\left(\frac{\log\alpha}{w}-a\right)}dt$$

$$= \frac{1}{\log\alpha - aw}.$$

This result will be useful to find the following functions. We obtain

$$S_\alpha[\sin at] = \frac{aw}{(\log\alpha)^2+(aw)^2},$$

$$S_\alpha[\cos at] = \frac{\log\alpha}{(\log\alpha)^2+(aw)^2},$$

$$S_\alpha[e^{at}\sin bt] = \frac{bw}{(\log\alpha-aw)^2+(bw)^2} \text{ and}$$

$$S_\alpha[e^{at}\cos bt] = \frac{\log\alpha-aw}{(\log\alpha-aw)^2+(bw)^2}.$$

(v) If $g(t) = t^n e^{at}, n \geq 0$, then

$$S_\alpha[g(t)] = S_\alpha[t^n e^{at}] = \frac{1}{w}\int_0^\infty t^n e^{-t\left(\frac{\log\alpha-aw}{w}\right)}dt$$

$$= \frac{n!w^n}{(\log\alpha-aw)^{n+1}}.$$

III. APPLICATIONS OF α-MODIFIED SUMUDU TRANSFORM

In this section, α-modified Sumudu transform is applied to find the solutions of real-life problems.

3.1 Application to Population Growth Problem

The doubling time of a population of flies is eight days. If there are initially 100 flies, then we will consider the population of flies in 17 days.

This problem can be written in mathematical form as:

$$\frac{dQ(t)}{dt} = KQ(t), \quad (8)$$

where K is the constant of proportionality, Q denotes the amount of flies population at time t and Q_0 is the initial amount of flies population at $t = 0$.

We take the α-modified Sumudu transform to (8), we get

$$\frac{\log\alpha}{w}S_\alpha[Q(t)] - \frac{1}{w}Q(0) = KS_\alpha[Q(t)].$$

At $t = 0$, $Q = Q_0 = 100$, we have

$$S_\alpha[Q(t)] = \frac{100}{\log\alpha - Kw}.$$

The inverse α-modified Sumudu transform of this equation yields the solution. We obtain

$$Q(t) = 100e^{Kt}. \quad (9)$$

When $t = 8$, $Q = 2Q_0 = 200$, then (9) becomes

$$K = \frac{\log_e 2}{8} \approx 0.0866.$$

We required Q when $t = 17$, then (9) becomes

$$Q(17) = 100e^{(0.0866)17} \approx 435.88 \text{ flies}.$$

3.2 Application to Decay Problem

Suppose that a substance has a half-life of eight days. If there are 40 grams present now, then we consider the amount of substance left after three days.

This problem can be written in mathematical form as:

$$\frac{dQ(t)}{dt} = -KQ(t), \quad (10)$$

where Q denotes the amount of substance at any time t, Q_0 is the initial amount of the substance at time $t = 0$ and K is the constant of proportionality.

Taking the α-modified Sumudu transform to (10), we get

$$\frac{\log\alpha}{w} S_\alpha[Q(t)] - \frac{1}{w} Q(0) = -KS_\alpha[Q(t)].$$

At $t = 0$, $Q = Q_0 = 40$, we have

$$S_\alpha[Q(t)] = \frac{40}{\log\alpha + Kw}.$$

The inverse α-modified Sumudu transform of this equation leads to the solution. We obtain

$$Q(t) = 40e^{-Kt}. \quad (11)$$

When $t = 8$, $Q = \frac{1}{2}Q_0 = 20$, we get $K \approx 0.0866$.

We required Q when $t = 3$, then (11) becomes

$$Q(3) = 40e^{(-0.0866)3} \approx 30.85 \text{ grams}.$$

There are 30.85 grams of the substance remaining after three days.

3.3 Applications to Mechanics

A particle Q of mass 4 grams moves on the X-axis and is attracted towards origin O with a force numerically equal to 16X. If it is initially at rest at $X = 10$, then we consider its position at any subsequent time assuming

(i) No other force acts,
(ii) A damping force numerically equal to 16 times the instantaneous velocity acts.
(iii) By Newton's law, the equation of motion of the particle is

$$4\frac{d^2X}{dt^2} = -16X \quad \text{(or)} \quad \frac{d^2X}{dt^2} + 4X = 0 \quad (12)$$

with initial conditions $X(0) = 10$ and $X'(0) = 0$.

Applying the α-modified Sumudu transform on both sides of (12), we get

$$\frac{(\log\alpha)^2}{w^2} S_\alpha[X(t)] - \frac{\log\alpha}{w} X(0) - \frac{1}{w} X'(0) + 4S_\alpha[X(t)] = 0.$$

Using the initial conditions, we have

$$S_\alpha[X(t)] = \frac{10\log\alpha}{(\log\alpha)^2 + 4w^2}.$$

The inverse α-modified Sumudu transform of this equation gives the solution.

$$X(t) = 10\cos 2t.$$

(i) In this case, the equation of motion of the particle is

$$4\frac{d^2X}{dt^2} = -16X - 16\frac{dX}{dt} \quad \text{(or)}$$

$$\frac{d^2X}{dt^2} + 4\frac{dX}{dt} + 4X = 0 \quad (13)$$

with initial conditions $X(0) = 10$ and $X'(0) = 0$.

Taking the α-modified Sumudu transform to (13), we get

$$\frac{(\log\alpha)^2}{w^2} S_\alpha[X(t)] - \frac{\log\alpha}{w} X(0) - \frac{1}{w} X'(0) + \frac{4\log\alpha}{w} S_\alpha[X(t)] - \frac{4}{w} X(0) + 4S_\alpha[X(t)] = 0.$$

Using the initial conditions, we have

$$S_\alpha[X(t)] = \frac{10}{\log\alpha + 2w} + \frac{20w}{(\log\alpha + 2w)^2}.$$

We take the inverse α-modified Sumudu transform to this equation. The solution is

$$X(t) = 10e^{-2t} + 20te^{-2t}.$$

3.4 Application to Electrical Circuits

An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of 0.02 farads are connected in series with an e.m.f of E volts. At $t = 0$, the charge on the capacitor and current in the circuit are zero. We will find the charge and current at any time t > 0 if $E = 300$ volts.

Let Q and I be the instantaneous charge and current respectively at time t.

By using the second Kirchhoff's law, we have

$$2\frac{dI}{dt} + 16I + \frac{Q}{0.02} = 300,$$

$$\frac{d^2Q}{dt^2} + 8\frac{dQ}{dt} + 25Q = 150 \qquad (14)$$

with initial conditions $Q(0) = 0$, $I(0) = Q'(0) = 0$.

Taking the α-modified Sumudu transform on both sides of (14), we get

$$\frac{(\log\alpha)^2}{w^2} S_\alpha[Q(t)] - \frac{\log\alpha}{w} Q(0) - \frac{1}{w} Q'(0)$$

$$+ \frac{8\log\alpha}{w} S_\alpha[Q(t)] - \frac{8}{w} Q(0) + 25S_\alpha[Q(t)] = \frac{150}{\log\alpha}.$$

Using the initial conditions, we have

$$S_\alpha[Q(t)] = \frac{6}{\log\alpha} - \frac{6\log\alpha + 24w}{(\log\alpha + 4w)^2 + 9w^2}$$

$$- \frac{24w}{(\log\alpha + 4w)^2 + 9w^2}.$$

We take the inverse α-modified Sumudu transform of this equation, the charge is

$$Q(t) = 6 - 6e^{-4t}\cos 3t - 8e^{-4t}\sin 3t.$$

Then the current is

$$I = \frac{dQ}{dt} = 50e^{-4t}\sin 3t.$$

CONCLUSION

The α-modified Sumudu transform method is simple method to solve the solutions of ordinary differential equations. The solving real-life problems by using analytical method and using α-modified Sumudu transform method have the same answer. In addition to, the solving ordinary differential equations expressed in this paper, solving partial differential equations can also be considered by using α-modified Sumudu transform method. This paper demonstrates that α-modified Sumudu transform is not a mere formula and an essential tool in solving the solutions of ordinary differential equations and partial differential equations.

REFERENCES

[1] Belgacem, F. B. M., Karaballi, A. A. and Kalla, L. S., 2003, "Analytical Investigations of the Sumudu Transform and Applications to Integral Production Equations", Mathematical Problems in Engineering, Vol. 3, pp. 103-118.

[2] Debnath, L. and Bhatta, D., 2006, "Integral Transforms and Their Applications", second Edition, A Chapman & Hall Book/CRC Press, New York.

[3] Eltayeb, H. and Kilicman, A., 2010, "A Note on the Sumudu Transforms and Differential Equations", Applied Mathematical Sciences, Vol. 4, No. 22, pp. 1089-1098.

[4] Elzaki, T. M., Ezaki, S. M. and Elnour, E. A., 2012, "Applications of New Transform: Elzaki Transform to Mechanics, Electrical Circuits and Beams Problems", Global Journal of Mathematical Sciences: Theory and Practical, Vol. 1, No. 1, pp 25-34.

[5] Kilicman, A., Eltayeb, H. and Ismail, M. R., 2012, "A Note on Integral Transforms and Differential Equations", Malaysian Journal of Mathematical Sciences, Vol. 6, pp. 1-18.

[6] Watugula, G. K., 1993, "Sumudu Transform: A New Integral Transform to Solve Differential Equations and Control Engineering Problems", International Journal of Mathematical Education in Sciences and Technology, Vol. 24, pp. 35-43.