

Solution of Three-Dimensional Mboctara Equation by Triple Laplace Transform

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Abstract- In this paper, the definitions of triple Laplace transforms and inverse triple Laplace transforms are firstly presented. And then, the properties of triple Laplace transforms are described. In addition, triple Laplace transforms of some elementary functions are expressed. Moreover, the triple Laplace transforms of partial differential derivatives are calculated. Finally, triple Laplace transform is utilized to solve Mboctara partial differential equations.

Indexed Terms- Triple Laplace Transform, Mboctara Equations

I. INTRODUCTION

Triple Laplace transform method can be used for solving differential and integral equations. In this paper, we will concern with Mboctara partial differential equations by using the triple Laplace transform method. In Section 1, definitions of triple Laplace transform and its properties are described. In Section 2, the triple Laplace transform method is applied to solve the derivatives of partial differential equations. In Section 3, the proposed triple Laplace transform method is utilized to solve three-dimensional Mboctara equations.

1.1 Definition ^[4]

The triple Laplace transform of a function $f(x, y, t)$ of three variables x, y and t defined in the first octant of the xyt -space is defined by the triple integral in the form

$$\bar{\bar{\bar{f}}}(p, q, s) = F(p, q, s) = L_3[f(x, y, t)] = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} f(x, y, t) dx dy dt, \quad (1)$$

provided the integral exists.

1.2 Definition ^[4]

The inverse triple Laplace transform $L_3^{-1}[\bar{\bar{\bar{f}}}(p, q, s)] = f(x, y, t)$ is defined by the complex triple integral formula

$$L_3^{-1}[\bar{\bar{\bar{f}}}(p, q, s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} dp \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{qy} dq \frac{1}{2\pi i} \int_{e-i\infty}^{e+i\infty} e^{st} F(p, q, s) dt, \quad (2)$$

where $\bar{\bar{\bar{f}}}(p, q, s)$ must be an analytic function for all p, q, s in the region defined by $\text{Re } p \geq 0, \text{ Re } q \geq 0, \text{ Re } s \geq 0$, for some c, d, e are real constants.

1.3 Definition ^[4]

The function $f(x, y, t)$ is said to be exponential order a, b and c on $0 \leq x \leq \infty, 0 \leq y \leq \infty$ and $0 \leq t \leq \infty$ if there exists a positive constant K such that for all $x > X, y > Y$ and $t > T$, then $|f(x, y, t)| \leq K e^{-(ax+by+ct)}$.

II. SOME PROPERTIES OF TRIPLE LAPLACE TRANSFORM

In this section, some basic properties of triple Laplace transform are expressed.

2.1 Theorem

If $f(x, y, t)$ and $g(x, y, t)$ are two functions of x, y and t such that $L_3[f(x, y, t)] = \bar{\bar{\bar{f}}}(p, q, s)$ and $L_3[g(x, y, t)] = \bar{\bar{\bar{g}}}(p, q, s)$, then $L_3[\alpha f(x, y, t) + \beta g(x, y, t)] = \alpha L_3[f(x, y, t)] + \beta L_3[g(x, y, t)]$,

where α and β are constants.

Proof: See[8].

2.2 Theorem

If $f(x, y, t) = f(x)g(y)h(t)$, then

$$L_3[f(x, y, t)] = F(p)G(q)H(s).$$

Proof: See[4].

2.3 Theorem

If the function $f(x, y, t)$ is a continuous function in every finite intervals $(X, 0, 0), (0, Y, 0), (0, 0, T)$ and of exponential order $e^{ax+by+ct}$, then the triple Laplace transform of $f(x, y, t)$ exists for all p, q, s provided $\text{Re } p \geq a, \text{Re } q \geq b, \text{Re } s \geq c$.

Proof: See[4].

III. TRIPLE LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

In this section, triple Laplace transforms of some elementary functions are described.

(i) If $f(x, y, t) = 1$ for all $x > 0, y > 0$ and $t > 0$, then

$$L_3[1] = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} dx dy dt = \int_0^\infty e^{-px} dx \int_0^\infty e^{-qy} dy \int_0^\infty e^{-st} dt = \frac{1}{pqs}.$$

(ii) If $f(x, y, t) = (xyt)^n$ for all $x > 0, y > 0$ and $t > 0$, then

$$L_3[(xyt)^n] = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} (xyt)^n dx dy dt$$

$$= \int_0^\infty e^{-px} x^n dx \int_0^\infty e^{-qy} y^n dy \int_0^\infty e^{-st} t^n dt = \frac{(n!)^3}{(pqs)^{n+1}},$$

where n is a positive integer.

(iii) If $f(x, y, t) = e^{(ax+by+ct)}$ for all x, y and t , then

$$L_3[e^{(ax+by+ct)}] = \int_0^\infty e^{-(p-1)x} dx \int_0^\infty e^{-(q-1)y} dy \int_0^\infty e^{-(s-1)t} dt$$

$$= \frac{1}{(p-a)(q-b)(s-c)}.$$

$$\text{Similarly, } L_3[e^{i(ax+by+ct)}] = \frac{1}{(p-ia)(q-ib)(s-ic)}.$$

Thus, one can obtain

$$L_3[e^{i(ax+by+ct)}] = \frac{(pqs - abs - pbc - qac) + i(pbs + aqs + pqc - abc)}{(p^2 + a^2)(q^2 + b^2)(s^2 + c^2)}.$$

Consequently,

$$L_3[\cos(ax + by + ct)] = \frac{(pqs - abs - pbc - qac)}{(p^2 + a^2)(q^2 + b^2)(s^2 + c^2)}.$$

$$L_3[\sin(ax + by + ct)] = \frac{(pbs + aqs + pqc - abc)}{(p^2 + a^2)(q^2 + b^2)(s^2 + c^2)}.$$

$$L_3[\cosh(ax + by + ct)] = \frac{1}{2} \left[\frac{1}{(p-a)(q-b)(s-c)} + \frac{1}{(p+a)(q+b)(s+c)} \right].$$

$$L_3[\sinh(ax + by + ct)] = \frac{1}{2} \left[\frac{1}{(p-a)(q-b)(s-c)} - \frac{1}{(p+a)(q+b)(s+c)} \right].$$

IV. TRIPLE LAPLACE TRANSFORMS FOR PARTIAL DIFFERENTIAL DERIVATIVES

In this section, triple Laplace transforms of partial differential derivatives are calculated.

4.1 Example

$$\text{If } \bar{\bar{\bar{u}}}(p, q, s) = L_3[u(x, y, t)], \bar{\bar{u}}_1(q, s) = L_2[u(0, y, t)],$$

$$\text{then } L_3\left[\frac{\partial u}{\partial x}\right] = p\bar{\bar{\bar{u}}}(p, q, s) - \bar{\bar{u}}_1(q, s).$$

From the definition of triple Laplace transform, we have

$$L_3\left[\frac{\partial u}{\partial x}\right] = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial}{\partial x} u(x, y, t) dx dy dt$$

$$= \int_0^\infty \int_0^\infty e^{-(qy+st)} \left[\int_0^\infty e^{-px} \frac{\partial}{\partial x} u(x, y, t) dx \right] dy dt$$

$$= \int_0^\infty \int_0^\infty e^{-(qy+st)} \left\{ \left[e^{-px} u(x, y, t) \right]_0^\infty - \int_0^\infty u(x, y, t) (-p) e^{-px} dx \right\} dy dt$$

$$\begin{aligned}
&= -\int_0^\infty \int_0^\infty e^{-(qy+st)} u(0, y, t) dy dt + p \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} u(x, y, t) dx dy dt \\
&= -L_2 [u(0, y, t)] + pL_3 [u(x, y, t)] \\
&= -\bar{\bar{u}}_1(q, s) + p\bar{\bar{u}}(p, q, s).
\end{aligned}$$

4.2 Example

If $\bar{\bar{u}}_1(q, s) = L_2 [u_x(0, y, t)]$, then

$$L_3 \left[\frac{\partial^2 u}{\partial x^2} \right] = p^2 \bar{\bar{u}}(p, q, s) - p\bar{\bar{u}}_1(q, s) - \bar{\bar{u}}_4(q, s).$$

From the definition of triple Laplace transform, we have

$$\begin{aligned}
L_3 \left[\frac{\partial^2 u}{\partial x^2} \right] &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial^2}{\partial x^2} u(x, y, t) dx dy dt \\
&= \int_0^\infty \int_0^\infty e^{-(qy+st)} \left[\int_0^\infty e^{-px} \frac{\partial^2}{\partial x^2} u(x, y, t) dx \right] dy dt \\
&= -\int_0^\infty \int_0^\infty e^{-(qy+st)} \frac{\partial}{\partial x} u(0, y, t) dy dt \\
&\quad + p \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial}{\partial x} u(x, y, t) dx dy dt \\
&= -L_2 [u_x(0, y, t)] + pL_3 \left[\frac{\partial}{\partial x} u(x, y, t) \right] \\
&= -L_2 [u_x(0, y, t)] + p \{ -L_2 [u(0, y, t)] + pL_3 [u(x, y, t)] \} \\
&= -L_2 [u_x(0, y, t)] - pL_2 [u(0, y, t)] + p^2 L_3 [u(x, y, t)] \\
&= -\bar{\bar{u}}_4(q, s) - p\bar{\bar{u}}_1(q, s) + p^2 \bar{\bar{u}}(p, q, s).
\end{aligned}$$

4.3 Example

If $L [u_x(x, 0, 0)] = p\bar{u}(p, 0, 0) - u(0, 0, 0)$, then

$$\begin{aligned}
L_3 \left[\frac{\partial^3 u(x, y, t)}{\partial x \partial y \partial t} \right] &= pqs\bar{\bar{u}}(p, q, s) - pq\bar{\bar{u}}(p, q, 0) - ps\bar{\bar{u}}(p, 0, s) \\
&\quad - qs\bar{\bar{u}}(0, q, s) + p\bar{u}(p, 0, 0) + q\bar{u}(0, q, 0) + s\bar{u}(0, 0, s) - u(0, 0, 0).
\end{aligned}$$

From the definition of triple Laplace transform, we have

$$\begin{aligned}
L_3 \left[\frac{\partial^3 u}{\partial x \partial y \partial t} \right] &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial^3 u}{\partial x \partial y \partial t} u(x, y, t) dx dy dt \\
&= \int_0^\infty \int_0^\infty e^{-(px+qy)} \left[\int_0^\infty e^{-st} \frac{\partial^3}{\partial x \partial y \partial t} u(x, y, t) dt \right] dx dy \\
&= -\int_0^\infty \int_0^\infty e^{-(px+qy)} \frac{\partial^2}{\partial x \partial y} u(x, y, 0) dx dy \\
&\quad + s \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial^2}{\partial x \partial y} u(x, y, t) dx dy dt \\
&= -\int_0^\infty e^{-px} \left\{ \left[e^{-qy} \frac{\partial}{\partial x} u(x, y, 0) \right]_0^\infty - \int_0^\infty \frac{\partial}{\partial x} u(x, y, 0) (-q) e^{-qy} dy \right\} dx \\
&\quad + s \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial^2}{\partial x \partial y} u(x, y, t) dx dy dt \\
&= \int_0^\infty e^{-px} \frac{\partial}{\partial x} u(x, 0, 0) dx - q \int_0^\infty \int_0^\infty e^{-(px+qy)} \frac{\partial}{\partial x} u(x, y, 0) dx dy \\
&\quad + s \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial^2}{\partial x \partial y} u(x, y, t) dx dy dt \\
&= -u(0, 0, 0) + pL [u(x, 0, 0)] - q \int_0^\infty \int_0^\infty e^{-(px+qy)} \frac{\partial}{\partial x} u(x, y, 0) dx dy \\
&\quad + s \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial^2}{\partial x \partial y} u(x, y, t) dx dy dt \\
&= -u(0, 0, 0) + pL [u(x, 0, 0)] + qL [u(0, y, 0)] - pqL_2 [u(x, y, 0)] \\
&\quad + s \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)} \frac{\partial^2}{\partial x \partial y} u(x, y, t) dx dy dt \\
&= -u(0, 0, 0) + pL [u(x, 0, 0)] + qL [u(0, y, 0)] - pqL_2 [u(x, y, 0)] \\
&\quad + s \int_0^\infty \int_0^\infty \int_0^\infty e^{-(qy+st)} \left[-\frac{\partial}{\partial y} u(0, y, t) + p \int_0^\infty e^{-px} \frac{\partial}{\partial y} u(x, y, t) dx \right] dy dt \\
&= pqsL_3 [u(x, y, t)] - pqL_2 [u(x, y, 0)] - psL_2 [u(x, 0, t)] \\
&\quad - qsL_2 [u(0, y, t)] + pL [u(x, 0, 0)] + qL [u(0, y, 0)] \\
&\quad + sL [u(0, 0, t)] - u(0, 0, 0) \\
&= pqs\bar{\bar{u}}(p, q, s) - pq\bar{\bar{u}}(p, q, 0) - ps\bar{\bar{u}}(p, 0, s) \\
&\quad - qs\bar{\bar{u}}(0, q, s) + p\bar{u}(p, 0, 0) + q\bar{u}(0, q, 0) \\
&\quad + s\bar{u}(0, 0, s) - u(0, 0, 0).
\end{aligned} \tag{3}$$

V. APPLICATIONS

In this section, triple Laplace transform method is applied to solve three-dimensional Mboctara partial differential equations.

5.1 Example

We consider the third-order partial differential equation

$$\partial_{xyt} u(x, y, t) + u(x, y, t) = 0, \quad (4)$$

subject to the boundaries and initial conditions

$$\begin{aligned} u(x, y, 0) &= e^{x+y}, \quad u(x, 0, t) = e^{x-t}, \\ u(0, y, t) &= e^{y-t}, \quad u(x, y, 1) = e^{x+y-1}. \end{aligned} \quad (5)$$

Taking the triple Laplace transform of (4), we have

$$\begin{aligned} L_3 [\partial_{xyt} u(x, y, t)] + L_3 [u(x, y, t)] &= 0, \\ -U(0, 0, 0) + pL[u(x, 0, 0)] + qL[u(0, y, 0)] \\ + sL[u(0, 0, t)] - qsL_2[u(0, y, t)] - pqL_2[u(x, y, 0)] \\ - psL_2[u(x, 0, t)] + pqsU(p, q, s) + U(p, q, s) &= 0, \\ (pqs + 1)U(p, q, s) &= U(0, 0, 0) - pL[u(x, 0, 0)] \\ - qL[u(0, y, 0)] - rL[u(0, 0, t)] + pqL_2[u(x, y, 0)] \\ + psL_2[u(x, 0, t)] + qsL_2[u(0, y, t)]. \end{aligned}$$

By using the boundaries and initial conditions, we have

$$\begin{aligned} (pqs + 1)U(p, q, s) &= 1 - pL[e^x] - qL[e^y] \\ - sL[e^{-t}] + psL_2[e^{x-t}] + qsL_2[e^{y-t}] + pqL_2[e^{x+y}] \\ &= 1 - \frac{p}{(p-1)} - \frac{q}{(q-1)} + \frac{ps}{(p-1)(s+1)} \\ &\quad - \frac{s}{(s+1)} + \frac{qs}{(q-1)(s+1)} + \frac{pq}{(p-1)(q-1)} \\ &= \frac{1}{(p-1)(q-1)(s+1)} \{ (p-1)(q-1) + ps(q-1) \\ &\quad + (s+1)[-pq + q + p] + qs(p-1) \} \\ U(p, q, s) &= \frac{1}{(p-1)(q-1)(s+1)}. \end{aligned} \quad (6)$$

By taking the inverse triple Laplace transform of (6), we obtain

$$u(x, y, t) = e^{x+y-t}.$$

5.2 Example

We consider the nonhomogeneous third-order Mbactara partial differential equation

$$\partial_{xyt} u(x, y, t) + u(x, y, t) = 3e^{-x-2y+t}, \quad (7)$$

subject to the boundaries and initial conditions,

$$\left. \begin{aligned} u(x, y, 0) &= e^{-x-2y}, \quad u(x, 0, 0) = e^{-x}, \\ u(x, 0, t) &= e^{-x+t}, \quad u(0, y, 0) = e^{-2y}, \\ u(0, y, t) &= e^{-2y+t}, \quad u(0, 0, z) = e^t. \end{aligned} \right\} \quad (8)$$

Taking the triple Laplace transform of (7), we have

$$\begin{aligned} L_3 [\partial_{xyt} u(x, y, t)] + L_3 [u(x, y, t)] &= 3L_3 [e^{-x-2y+t}], \\ -U(0, 0, 0) + pL[u(x, 0, 0)] + qL[u(0, y, 0)] - pqL_2[u(x, y, 0)] \\ + sL[u(0, 0, t)] - qsL_2[u(0, y, t)] - psL_2[u(x, 0, t)] \\ + pqsU(p, q, s) + U(p, q, s) &= 3L_3 [e^{-x-2y+t}], \\ (pqs + 1)U(p, q, s) &= U(0, 0, 0) - pL[u(x, 0, 0)] \\ - qL[u(0, y, 0)] + pqL_2[u(x, y, 0)] - sL[u(0, 0, t)] \\ + qsL_2[u(0, y, t)] + psL_2[u(x, 0, t)] + 3L_3 [e^{-x-2y+t}]. \end{aligned}$$

By using the boundaries and initial conditions, we have

$$\begin{aligned} (pqs + 1)U(p, q, s) &= 1 - pL[e^{-x}] - qL[e^{-2y}] \\ + pqL_2[e^{-x-2y}] - sL[e^t] + qsL_2[e^{-2y+t}] \\ + psL_2[e^{-x+t}] + 3L_3 [e^{-x-2y+t}] \\ &= 1 + \frac{pq}{(p+1)(q+2)} - \frac{s}{(s-1)} + \frac{ps}{(p+1)(s-1)} - \frac{q}{(q+2)} \\ &\quad + \frac{qs}{(q+2)(s-1)} - \frac{3}{(p+1)(q+2)(s-1)} - \frac{p}{(p+1)} \\ &= \frac{1}{(p+1)(q+2)(s-1)} \{ (qs - q + 2s - 2) \\ &\quad - q(s-1) - 2s(p+1) + pqs + 2ps + 3 \} \\ U(p, q, s) &= \frac{1}{(p+1)(q+2)(s-1)}. \end{aligned} \quad (9)$$

By taking the inverse triple Laplace transform of (9), we obtain

$$u(x, y, t) = e^{-x-2y+t}.$$

5.3 Example

We consider the nonhomogeneous third-order Mbactara partial differential equation

$$\partial_{xyt} u(x, y, t) + u(x, y, t) = \cos x \cos y \cos t - \sin x \sin y \sin t, \quad (10)$$

subject to the boundaries and initial conditions

$$\begin{aligned}
u(x, y, 0) &= \cos x \cos y, \quad u(x, 0, 0) = \cos x, \\
u(x, 0, t) &= \cos x \cos t, \quad u(0, y, 0) = \cos y, \\
u(0, y, t) &= \cos y \cos t, \quad u(0, 0, t) = \cos t.
\end{aligned} \quad (11)$$

Taking the triple Laplace transform of (10), we have

$$\begin{aligned}
L_3[\partial_{xyt} u(x, y, t)] + L_3[u(x, y, t)] &= L_3[\cos x \cos y \cos t] \\
&\quad - L_3[\sin x \sin y \sin t], \\
-U(0, 0, 0) + pL[u(x, 0, 0)] + qL[u(0, y, 0)] \\
&\quad - pqL_2[u(x, y, 0)] + sL[u(0, 0, t)] - qsL_2[u(0, y, t)] \\
&\quad - psL_2[u(x, 0, t)] + U(p, q, s) + pqsU(p, q, s) \\
&= L_3[\cos x \cos y \cos t] - L_3[\sin x \sin y \sin t], \\
(pqs + 1)U(p, q, s) &= U(0, 0, 0) - pL[u(x, 0, 0)] \\
&\quad - qL[u(0, y, 0)] + pqL_2[u(x, y, 0)] - sL[u(0, 0, t)] \\
&\quad + qsL_2[u(0, y, z)] + psL_2[u(x, 0, t)] \\
&\quad + L_3[\cos x \cos y \cos t] - L_3[\sin x \sin y \sin t].
\end{aligned}$$

By using the boundaries and initial conditions, we have

$$\begin{aligned}
(pqs + 1)U(p, q, s) &= 1 - pL[\cos x] - qL[\cos y] \\
&\quad + pqL_2[\cos x \cos y] - sL[\cos t] \\
&\quad + qsL_2[\cos y \cos t] + psL_2[\cos x \cos t] \\
&\quad + L_3[\cos x \cos y \cos t] - L_3[\sin x \sin y \sin t] \\
&= \frac{1}{(p^2 + 1)(q^2 + 1)(s^2 + 1)} \{ (p^2 + 1)(q^2 + 1) \\
&\quad - q^2(s^2 + 1) - s^2(p^2 + 1) + p^2q^2s^2 + p^2s^2 + pqs - 1 \} \\
U(p, q, s) &= \frac{pqs}{(p^2 + 1)(q^2 + 1)(s^2 + 1)}. \quad (12)
\end{aligned}$$

By taking the inverse triple Laplace transform of (12), we obtain

$$u(x, y, t) = \cos x \cos y \cos t.$$

CONCLUSION

In this paper, the triple Laplace transform method has been applied for finding the exact solution of the Mboctara partial differential equations with initial conditions and boundary conditions. The result obtained by this triple Laplace transform method is found matched with the exact solution obtained by other existing methods.

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