© JUL 2025 | IRE Journals | Volume 9 Issue 1 | ISSN: 2456-8880

Applying Two-Dimensional Differential Transform Method to Partial Differential Equations

HNIN EI PHYU[1], KHIN THAN SINT[2]
[1,2] Department of Mathematics, University of Mandalay

Abstract- ***In this paper, definitions of two-dimensional differential transform and inverse differential transform are firstly described. And then, properties of two-dimensional differential transform are expressed. After that, two-dimensional differential transform is utilized to solve the initial value problems for linear homogeneous and inhomogeneous two-dimensional partial differential equations with constant coefficients and variable coefficients. Finally, two-dimensional differential transform is applied to solve the initial value problems for two-dimensional nonlinear partial differential equations***

Index Terms- Partial Differential Equations, Two-Dimensional Differential Transforms, Applications

I. INTRODUCTION

The differential transform method can be used for solving initial value problems for differential equations and integral equations. In this paper, we will be concerned with linear and nonlinear partial differential equations by using two-dimensional differential transform method.

In Section 1, the definitions of differential transform are expressed. In Section 2, its properties are presented. In Section 3, linear homogeneous two-dimensional partial differential equations with constant coefficients and variable coefficients are solved by using two-dimensional differential transform method. In Section 4, two-dimensional nonlinear partial differential equations with constant coefficients and variable coefficients are solved by applying two-dimensional differential transform method

1.1 Definition [2]

Two-dimensional differential transform of function $u(x, y)$ is defined as follows:

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial y^h} u(x, y)\right]_{(x,y)=(0,0)} . \quad (1)$$

In (1), $u(x, y)$ is the original function and $U(k,h)$ is the transformed function, which is called T-function in brief.

1.2 Definition[2]

Differential inverse transform of $U(k,h)$ is defined as follows:

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k y^h. \quad (2)$$

In fact, from (1) into (2), we obtain

$$u(x, y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial y^h} u(x,y)\right]_{(x,y)=(0,0)} x^k y^h, \quad (3)$$

which implies that the concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion. In this study we use the lower-case letters to represent the original functions and upper-case letters to stand for the transformed functions (T-functions).

II. SOME PROPERTIES OF TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM

Theorem 1

If $z(x,y)=u(x,y)\pm v(x,y)$, the

$Z(k,h)=U(k,h)\pm V(k,h)$.

Proof: See [Khatib, 2016].

Theorem 2

If $z(x,y)=\alpha u(x,y)$, then $Z(k,h)=\alpha U(k,h)$, where α is a constant.Proof: See [Khatib, 2016].

Theorem3

If $z(x,y)=\dfrac{\partial u(x,y)}{\partial x}$, th $Z(k,h)=(k+1)U(k+1,h)$.

Proof: See [Khatib, 2016].

Theorem 4

If $z(x,y)=\dfrac{\partial^{r+s}u(x,y)}{\partial x^r \partial y^s}$, then

$$Z(k,h)=(k+1)(k+2)\ldots(k+r)(h+1)(h+2)\ldots(h+s)U(k+r,h+s),$$

where r, s are positive integers.

Proof: See [Khatib, 2016].

Theorem 5

If $z(x,y)=u(x,y)v(x,y)$, then

$$Z(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s).$$

Proof: See [Khatib, 2016].

Theorem 6

If $z(x,y)=x^m y^n$, then

$Z(k,h)=\delta(k-m,h-n)=\delta(k-m)\delta(h-n)$, where

$$\delta(k-m)=\begin{cases}1, & k=m,\\ 0, & k\neq m,\end{cases}\quad \delta(h-n)=\begin{cases}1, & k=n,\\ 0, & k\neq n.\end{cases}$$

Proof: See [Khatib, 2016].

Theorem 7

If $z(x,y)=x^m e^{\lambda t}$, then $Z(k,h)=\delta(k-m)\dfrac{\lambda^h}{h!}$, where λ is a constant.

Theorem 8

If $z(x,y)=x^m\sin(\alpha t+\beta)$, then

$Z(k,h)=\delta(k-m)\dfrac{\alpha^h}{h!}\sin(\dfrac{h\pi}{2}+\beta)$, where α and β are constants.

Proof: See [Khatib, 2016].

III. SOLVING INITIAL VALUE PROBLEMS FOR LINEAR HOMOGENEOUS AND INHOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS

In this section, initial value problems for linear homogeneous and inhomogeneous partial differential equations are solved by using two-dimensional differential transform method.

3.1 Example 1

The initial value problem for linear homogenous Goursat problem is considered as follows:

$$u_{xt}=-u, \tag{4}$$

with initial conditions

$$u(x,0)=e^x, u(0,t)=e^{-t}, u(0,0)=1. \tag{5}$$

By applying the differential transform on (4), this leads to the following recurrence relation

$$(k+1)(h+1)U(k+1,h+1)=-U(k,h). \tag{6}$$

The initial conditions become

$$U(k,0)=\frac{1}{k!}, U(0,h)=\frac{(-1)^h}{h!}, U(0,0)=1. \tag{7}$$

Using recurrence relation (6) and conditions (7) for $k=0,1,2,\ldots,$ it is stated as follows:

$U(0,0)=1,\ U(0,1)=-1,\ U(0,2)=\dfrac{1}{2!},$

$U(0,3) = -\frac{1}{3!}$, $U(1,0) = 1$, $U(1,1) = -1$,

$U(1,2) = \frac{1}{2!}$, $U(1,3) = -\frac{1}{3!}$, $U(2,0) = \frac{1}{2!}$,

$U(2,1) = -\frac{1}{2!}$, $U(2,2) = \frac{1}{2!2!}$, $U(2,3) = -\frac{1}{2!3!}, \ldots$

Then, solution is stated as follows:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h, \\ = e^{x-t},$$

which is the exact solution.

3.2Example

The initial value problem for linear homogenous Goursat problem is considered as follows:

$$u_{xt} = 2u, \qquad (8)$$

with initial conditions

$$u(x,0) = e^x, u(0,t) = e^{2t}, u(0,0) = 1. \qquad (9)$$

By applying the differential transform on (8), this leads to the following recurrence relation

$$(k+1)(h+1)U(k+1,h+1) = 2U(k,h), \qquad (10)$$

The initial conditions become

$$U(k,0) = \frac{1}{k!}, U(0,h) = \frac{2^h}{h!}, U(0,0) = 1. \quad (11)$$

Using recurrence relation (10) and conditions (11) for $k = 0,1,2,\ldots,$ it is stated as follows:

$U(0,0) = 1, U(0,1) = 2, U(0,2) = \frac{2^2}{2!}$, $U(0,3) = \frac{2^3}{3!}$,

$U(1,0) = 1$, $U(1,1) = 2$, $U(1,2) = \frac{2^2}{2!}$, $U(1,3) = \frac{2^3}{3!}$,

$U(2,0) = \frac{1}{2!}$, $U(2,1) = \frac{2}{2!}$, $U(2,2) = \frac{2^2}{2!2!}$,

$U(2,3) = \frac{2^3}{2!3!}, \cdots.$

Then, solution is stated as follows:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h \\ = e^{x+2t},$$

which is the exact solution.

3.3Example

The initial value problem for linear inhomogenous Goursat problem is considered as follows:

$$u_{xt} = -u + t, \qquad (12)$$

with initial conditions

$$u(x,0) = e^{-x}, u(0,t) = t + e^t, u(0,0) = 1. \qquad (13)$$

By applying the differential transform on (12), this leads to the following recurrence relation

$$(k+1)(h+1)U(k+1,h+1) = U(k,h) + \delta(k,h-1), \qquad (14)$$

The initial conditions become

$$U(k,0) = \frac{(-1)^k}{k!}, U(0,h) = \frac{1}{h!} + \delta(h-1), U(0,0) = 1. \qquad (15)$$

Using recurrence relation (14) and conditions (15) for $k = 0,1,2,\ldots,$ it is stated as follows:

$U(0,0) = 1, U(0,1) = 2, U(0,2) = \frac{1}{2!}$, $U(0,3) = -\frac{1}{3!}$,

$U(1,0) = -1$, $U(1,1) = -1$, $U(1,2) = -\frac{1}{2!}$,

$U(1,3) = -\frac{1}{3!}$, $U(2,0) = \frac{1}{2!}$, $U(2,1) = \frac{1}{2!}$,

$U(2,2) = \frac{1}{2!2!}$, $U(2,3) = \frac{1}{2!3!}, \cdots.$

Then, solution is stated as follows:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h \\ = t + e^{t-x},$$

which is the exact solution.

3.4 Example

The initial value problem for linear inhomogenous Goursat problem is considered as follows:

$$u_{xt} = u + 8xt - 2x^2t^2, \quad (16)$$

with initial conditions

$$u(x,0) = e^x, u(0,t) = e^t, u(0,0) = 1. \quad (17)$$

By applying the differential transform on (16), this leads to the following recurrence relation

$$(k+1)(h+1)U(k+1,h+1) = U(k,h) + 8\delta(k-1,h-1) - 2\delta(k-2,h-2). \quad (18)$$

The initial conditions become,

$$U(k,0) = \frac{1}{k!}, U(0,h) = \frac{1}{h!}, U(0,0) = 1. \quad (19)$$

Using recurrence relation (18) and conditions (19) for $k = 0,1,2,\ldots,$ it is stated as follows:

$$U(0,0) = 1, U(0,1) = 1, U(0,2) = \frac{1}{2!}, \ U(0,3) = \frac{1}{3!},$$

$$U(1,0) = 1, \ U(1,1) = 1, \ U(1,2) = \frac{1}{2!}, U(1,3) = \frac{1}{3!},$$

$$U(2,0) = \frac{1}{2!}, \ U(2,1) = \frac{1}{2!}, U(2,2) = \frac{9}{2!2!},$$

$$U(2,3) = \frac{1}{2!3!}, \cdots.$$

Then, solution is stated as follows:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h = 2x^2t^2 + e^{x+t},$$

which is the exact solution.

IV. SOLVING INITIAL VALUE PROBLEMS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

In this section, initial value problems for nonlinear partial differential equations are solved by using two-dimensional differential transform method.

4.1 Example

The initial value problem for nonlinear nonhomogeneous Klein Gordon partial differential equation is considered as follows:

$$u_{tt} - u_{xx} + u^2 = -x\sin t + x^2\sin^2 t, \quad (20)$$

with initial conditions

$$u(x,0) = x, \ u_t(x,0) = x. \quad (21)$$

By applying the differential transform on (20), this leads to the following recurrence relation

$$(h+1)(h+2)U(k,h+2) - (k+1)(k+2)U(k+2,h) + \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)U(k-r,s)$$

$$= -\delta(k-1)\frac{\sin(\frac{h\pi}{2})}{h!} + \delta(k-2)\frac{2^h\sin(\frac{h\pi}{2})}{h!}. \quad (22)$$

From the first initial condition (21),

$$U(k,0) = \begin{cases} 1, & k = 1, \\ 0, & \text{otherwise,} \end{cases}$$

Also, from the second initial condition (21),

$$U(k,1) = \begin{cases} 1, & k = 1, \\ 0, & \text{otherwise,} \end{cases}$$

Using recurrence relation (22) and conditions for $k = 0,1,2,\ldots,$ it is stated as follows:

$$U(0,2) = 0, \ U(0,3) = 0, \ U(1,2) = 0, \ U(1,2) = -\frac{1}{3!},$$

$$U(3,2) = 0, \ U(1,4) = 0, \ U(3,3) = 0, \ U(1,5) = \frac{1}{5!}, \cdots.$$

Then, solution is stated as follows:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h,$$

$$= xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \cdots$$

$$= x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

$= x\sin t,$ which is the exact solution.

4.2 Example

The initial value problem for nonlinear partial differential equation is considered as follows:

$$u_{tt}(x,t) = u_{xx}(x,t) - u(x,t) - u^2(x,t) + xt + x^2t^2, \ x > 0, t > 0, \quad (23)$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = x, \qquad (24)$$

and boundary condition

$$u(0,t) = 0. \qquad (25)$$

By applying the differential transform on (23), this leads to the following recurrence relation

$$(h+1)(h+2)U(k,h+2) = (k+1)(k+2)$$
$$U(k+2,h) - U(k,h) -$$
$$\sum_{m=0}^{k}\sum_{n=0}^{h} U(m,h-n)U(k-m,n)$$
$$+\delta(k-1)\delta(h-1) + \delta(k-2)\delta(h-2),$$

$$U(k,h+2) = \frac{1}{(h+1)(h+2)}$$
$$[(k+1)(k+2)U(k+2,h) - U(k,h)$$
$$-\sum_{m=0}^{k}\sum_{n=0}^{h} U(r,h-s)U(k-r,s) + \delta(k-1)\delta(h-1)$$
$$+\delta(k-2)\delta(h-2)]. \qquad (26)$$

From the first initial condition (24),
$U(k,0) = 0,\ k = 0,1,\ldots.$

Also, from the second initial condition (24),

$$\sum_{k=0}^{\infty} U(k,0)x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left.\frac{d^k(x)}{dx^k}\right|_{x=0},$$

thus, $U(k,1) = \begin{cases} 1, & k = 1, \\ 0, & \text{otherwise.} \end{cases}$

From the boundary condition (25),
$U(0,h) = 0,\ k = 2,3,\ldots.$

Using recurrence relation (26) and conditions for $k = 0,1,2,\ldots,$ it is stated as follows: $U(0,2) = 0, U(1,2) = 0, U(2,2) = 0, \ldots.$

By continuing in this way, it can be described as follows:

Table 2

k/h	0	1	2	3	4	
0	0	0	0	0	0	…
1	0	1	0	0	0	…
2	0	0	0	0	0	0
3	0	0	0	0	0	…
4	0	0	0	0	0	…
⋮	…	…	…	…	…	…

Then, solution is stated as follows:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h$$
$$= xt,$$

which is the exact solution.

CONCLUSION

In this paper, the differential transform method has been applied for finding exact solution of linear and nonlinear equations with initial conditions and boundary conditions. According to the findings, the results revealed that the differential transform method (DTM) can be used not only in solving the linear and nonlinear partial differential equations with constant coefficients and variable coefficients but also in solving integral and integro-differential equations with constant coefficients and variable coefficients.

REFERENCES

[1] C. K. Chen, and S. H. Ho, "Solving Partial Differential Equations by Two-Dimensional Differential Transform Method", *Applied Mathematics and Computation*, Taiwan, Vol. 106, 1999, pp. 171-179.

[2] A. Khatib, "*Differential Transform Method for Differential Equations*", Master Thesis, Palestine Polytechnic University, Hebron, Palestine, 2016.

[3] H. Kishan, "*Differential Calculus*", Atlantic Publishers and Distributors (P) LTD, New Delhi, 2007.

[4] T. S. Priya, and N. Alima, "Differential Transform Method for Solving Linear and Homogeneous Equation with Variable Coefficients", *Journal of Emerging*

Technologies and Innovative Research, Vol. 5, Issue 4, 2016, pp. 424-426.

[5] K. R. Raslan, A. Biswas, and Z. Abu Sheer, "Differential Transform Method for Solving Partial Differential Equations with Variable Coefficients", *International Journal of Physical Sciences*, Saudi Arabia, Vol. 7, 2012, pp. 1412-1419.

[6] P. S. Sutkar, "Solution of Some Differential Equations by Using Differential Transform Method", *International Journal of Scientific and Innovative Mathematical Research*, Vol. 5, Issue 5, 2017, pp. 17-20.