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Distributed Delays in State and Control and Linear Delay Systems: Ideal Control

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Abstract- ***In order to prove the existence and uniqueness of optimum control for linear delay systems with distributed delays in state and control, the paper builds on the work of [6]. It is demonstrated that the optimal control is distinct and, assuming the system is reasonably controllable, takes the following form:***

$$U^* = \operatorname{sgn} c^T \int_{-h}^{0} (X(t_1, \hat{l} - s) dHl\hat{l} - s, s)$$

Index Terms : optimal control, linear Delays, Distributed delays, Null controllability.

I. INTRODUCTION

The study of optimal control by the control theorist is fast becoming fundamental as it presents the best amongst alternatives. As a result, the state of control theory is now presented in a comprehensive and thoughtful manner.

Presenting a systematic approach to the optimal control of linear systems with dispersed state and control delays is the aim of this work.

There are many definitions of optimal control derivable from controllability that is highly depends on the class of systems we are dealing with. However, it should be stressed that the result is achieved in minimum time. In the same view, the problem reaching the origin in time t, corresponds to null controllability. E.N. Chukwu [3] resolved the linear Neutral Functional systems time most effectively problem without delay in the control provided by

$$\frac{d}{dt} D(t, x_t) = L(t, x_t) + Bu(t)$$

where the target is an ongoing function in an n-dimensional Euclidean space, and the parameter set is in an m-dimensional unit cube.

This study provides essential and acceptable circumstances for optimal control to exist and be unique. However, in some cases stability of the systems under study have been established as in [2] and [7].

In a related work, Onwuatn [6] studied the system;

$$\dot{x}(t) = Ax(t) + \sum_{j=0}^{p} B_j x(t-j) + \sum_{j=0}^{p} D_j u(t-j)$$

and resolved the problem of optimal control of discrete systems in which he showed that, the degree to which a system can be controlled, then it is sufficient for it to be optimally controllable.

Eke and Nse [1] also diagnosed the ideal neutral control of the system with a non-linear base given by

$$\frac{d}{dt}(D(t, x_t) = A(x, t) + B(t)u(t)$$

They employ the method of the maximum principle of Pontryagin to be able to obtain the term of the most effect control. It is demonstrated that whenever the ideal control is present, then it is unique and bang - bang. Klamka [5] investigate the system, $\dot{x}(t) = A(t)x(t) + \int_{-h}^{0} [dsH(t,s)u(t+s)]$ and gave conditions for its relative controllability. Here in system [5], the delays are distributed. He showed that the system is relative controllable using square integrable controls if and if rank $W(t_0,t_1)=n$; where $W(t_0,t_1)$ is the controllability grammian.

II. MATERIALS AND METHODS

Notations and Preliminaries

In this study, we consider the linear delay system given by

$$\dot{x}(t)=L(t,x_t)+\int_{-h}^{0}[dsH(t,s)u(t+s)$$

(1.1)

$x(t)=E^n,\qquad u(t)=E^m,\qquad$ where

$$L(t,x_t)=\sum_{k=0}^{\infty}A_k x(t-W_k)+\int_{-h}^{0}A(t,\theta)x(t,\theta)$$

(1.2)

Satisfied almost everywhere on $[t_0,t_1]$. As n,m be integers that are positive, $E=(-\infty,+\infty)$ be the actual line. E^n is the n-dimensional Euclidean space with the The Euclidean norm is represented as $\|\cdot\|$; In E, j might be any interval. The typical Lebesque region of squared integrable functions (corresponding class of) from $J\rightarrow E^n$ be represented by $L_2(J,E^n)\cdot L_1([t_0,t_1],E^n)$ which denotes the region of integrable variables from $L_2(t_0,t_1)$ to $E^n\cdot Nn,m$ will be employed for gathering all $n\times m$ matrices with an appropriate norm. let $h>0$ be given. For a function $x:[t_0-h,t]\rightarrow E^n$ and $t\in[t_0,t_1]$, we use the symbol x_t to represent the variables on $[-h,0]$ defined by $x_t(s)=x(t+s)$ for $S\in[-h,0]$. The symbol $C=C([-h,0],E^n)$ represents the region of continuous variables mapping the interval $[-h,0]$, $h>0$ into E^n. Likewise, for variables, $U:[t_0-h,t_1]\rightarrow E^n$, $t\in[t_0,t_1]$, we use u_t to represent the variable on [-h,0] defined by $u(s)=u(t+s)$ for $S\in[-h,0]$. $x(t)\in C$; $u\in L_2([t_0,t_1],E^m)$; $L(t,\phi)$ is continuous in t and linear in ϕ;

$H(t,s)$ is an $n\times m$ matrix-valued variable that can be quantified in (t,s). We'll presume that $H(t,s)$ is of bounded-variation in S on [-h, 0] for each $t\in[t_0,t_1]$;

$A(t)\in L_1([t_0,t_1],Nn,m)$. The control-sets of interest in the follow-up are $B=L_2([t_0,t_1],Nn,m)$, $U\subseteq L_2([t_0,t_1],E^m)$ a bounded, finite segment of B that has zero inside of it with relation to B. If X and Y are linear spaces and $T:X\rightarrow Y$ is a mapping, We'll make use of the symbols D(T), R(T) and N(T) represents the null spaces, domain, and range of T respectively.

Definition 1.1: At time t, the full stale of system (1.1) is provided by

$$z(t)=\{x(t),x_t,U_t\}\qquad(1.3)$$

Definition 1.2: System (1.1) is relatively controllable on $[t_0,t_1]$ if for every initial complete state z(t_0) and every $x_1\in E^n$, there exists a control $U\in B$ such the system (1.1)'s equivalent path matches $x(t_1)=x_1$ whenever $x(t_0)=x_0$

Definition 1.3: System (1.1) is said to be relatively null controllable if in definition (1.2), the response $x(t)$ is achieved by the system $x(t_1)=0$

Definition 1.4: If System (1.1) is included in the permissible controls, it is considered optimally controllable $U\in B$, there exists a $U^*\in U$ such that the path of system (1.1) matches $x(t_1)=x_1$ in minimum time.

The solution of system (1.1) is of the form

$$x(t,t_0,\phi,u) = X(t_0,t_1)\phi(0) + \int_{t_0}^{t} X(t,\tau)[\int_{-h}^{0} dXH(t,s)U(\tau+s)]d\tau \quad (1.4)$$

where $X(\cdot,s) + X(t+\theta,s); -h < \theta < 0$

(1.5)

and $X(t,s)$ is the fundamental matrix solution of

$$\dot{x} = L(t,x_t) \quad (1.6)$$

satisfying $\dfrac{dX(t,s)}{dt} = L(t,X_t(\cdot,s))$ (1.7)

at most everywhere in (t,s) and

$$X(t,s) = \begin{cases} 0 & s-h \le t < s \\ I & t = s,\ I = identity \end{cases} \quad (1.8)$$

Now, we specify the $n \times m$ matrix of system controllability (1.1) by

$$W(t_0,t_1) = \int_{t_0}^{t_1}\left\{[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)][\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)]^T d\tau \quad (1.9)\right.$$

III.

where

$$\overline{H}(t,s) = \begin{cases} H(t,s) & for\ t \le t_1,\ s \in E \\ 0 & for\ t > t_1,\ s \in E \end{cases} \quad (1.10)$$

and the matrix transpose is indicated by T.

Definition 1.5: The set of Reachable $R(t_1,t_0)$ of system (1.1) is a subset of E^n given by

$$R(t_1,t_0)$$

$$\left\{\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)]u(\tau)d\tau, u \in U\right\} \quad (1.11)$$

Definition 1.6: It is claimed that System (1.1) is appropriate in E^n on $[t_0,t_1]$

$$\text{if } C^T[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)] = 0 \quad (1.12)$$

at most everywhere, $t \in [t_0,t_1]$, $c \in E^n$ implies that $c = 0$.

Definition 1.7: The attainable set $A(t)$ of system (1.1) is given by $A(t) = \{x(t,u); u \in U\}$ and is the collection of all potential system solutions. The optimal control problem seeks to identify an acceptable control U^* so that the outcome $x(t,\phi,u^*)$ of a certain system reaches a goal point in the shortest amount of time t^*. This is the best control and t^* the optimal time. The optimal question can thus be answered: u^* is an optimal control if there exist $t^* = \{t_{\inf} : A(t_1,x) \cap G(t,u) \neq 0$ for $t \ge t_1$ and $u^* = \{\min U : A(t,x) \cap G(t,u) \neq 0$ for some $t \ge t_1$. Let $z(t)$ be a continuous target of the general control system given by system (1.1), if there exists an admissible control $u \in U$ and a time $t \ge 0$ for which $x(t_1,u) = z(t)$, then there exists an optimal control, that is the solution hits the target in minimum time.

III. RESULTS

Here we state one proposition and one theorem for the relative controllability of system (1.1)

Proposition 2.1: The following statements are equivalent

(i) $W(t_0,t_1)$ is non-singular for each $t_1 > t_0$

(ii) System (1.1) is proper in E^n for each interval $[t_0,t_1]$

(iii) At each interval, System (1.1) is comparatively controlled $[t_0,t_1]$

Proof: $(i) \Rightarrow (ii)$

Let

$$W(t_0,t_1) = \int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)][\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)^T d\tau.$$

Define the operator $K : L_2([t_0,t_1],E^n) \to E^n$.

$$K(u) = \int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)][\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)]u(\tau)d\tau \quad (2.1)$$

From one Hilbert space to another, K is a continuous linear operator. Thus $R(k) \subset E^*$ is a proportional subspace and the relation is satisfied by its orthogonal complement $(R(k))' = N(k^*)$

(2.2)

where k^* is the adjoint of K. By the non – singularity of $W(t_0,t_1)$, The operator which is symmetric $KK^T = W(t_0,t_1)$ is positive definite, thus $\{R(k)' = \{0\}\}$ (2.3)

for any $c \in E^n$, $u \in L_2$; $<c,ku> = <k^*c,u>$

$$<c,ku> = <c\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)]u(\tau)d\tau \quad (2.4)$$

$$= \int_{t_0}^{t_1} C^T \int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)]u(\tau)d\tau \quad (2.5)$$

Thus k is given by

$$C \to C^T \int_{t}^{t_1}[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s); \tau \in [t_0,t_1]$$

$N(k^*)$ is therefore the set of all such $c \in E^n$ such that

$$C^T[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)] = 0 \quad (2.6)$$

at most points in $[t_0,t_1]$. since $\{N(k^*) = \{0\}\}$, These c are all corresponding to zero; that is $c = 0$. This establishes properness of system (1.1).

$(ii) \to (iii)$

We now demonstrate that, if system (1.1) is correct, it can be reasonably controlled at each interval $[t_0,t_1]$ Let $c \in E^n$, if system (1.1) is proper, then $C^T[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)] = 0$ at most which implies $c = 0$

Thus,

$$\int_{t_0}^{t_1} C^T[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)]u(\tau)d\tau = 0$$

For $u \in L_2$. Consequently, the sole vector perpendicular to the set

$$R(t_1,t_0) = \{\int_{t_0}^{t_1}[\int_{-h}^{0} X(t_1,\tau-s)dH(\tau-s,s)]u(\tau)d\tau; u \in L_2\} \quad (2.7)$$

is the zero vector. Hence $R(t_1,t_0) = 0$, that is, $R(t_1,t_0) = E^n$. this indicates that the system is somewhat manageable on Euclidean terms $[t_1,t_0]$

$(iii) \Rightarrow (i)$

We now demonstrate that the controllability grammian, if the system is sufficiently controllable $W = W(t_0,t_1)$ is non – singular

Assume for the purposes of this contradiction that W is singular. Then, an n-vector v exists such that $vWU^T = 0$. Then

$$\int_{t_0}^{t_1} \left\| v[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)] \right\|^2 d\tau = 0 \quad (2.8)$$

This implies that

$\left\| v[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)] \right\|^2 d\tau = 0$ at most everywhere. Hence

$v[\int_{-h}^{0} X(t_1,\tau-s)d\overline{H}(\tau-s,s)] = 0$,atmost everywhere for $t \in [t_0,t_1]$. The properness assumption is violated in this case because $v \neq 0$. This completes the proof.

Theorem 2.1
If and only if system (1.1) is reasonably controllable $0 \in \operatorname{int} R(t_0, t_1)$ for each $t_1 > t_0$.

Proof:
$R(t_0, t_1)$ is a convex, closed subset of E^n. Consequently, a point on the border of $R(t_0, t_1)$ suggests the presence of a support plane π of $R(t_0, t_1)$ through y_1. That is $C^T(y - y_1) \le 0$ for each $y \in R(t_0, t_1)$ where $c \neq 0$ is an outward normal to π. If u, is the control that goes with y_1, we have

$$C^T \int_{t_0}^{t_1} [\int_{-h}^{0} X(t_1, \tau, s) d\overline{H}(\tau - s, s)] u(\tau) d\tau \le C^T \int_{t_0}^{t_1} [\int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s)] u(\tau) d\tau \quad (2.9)$$

For each $u \in U$. This last inequality holds if and only if

$$C^T \int_{t_0}^{t_1} [\int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s)] u(\tau) d\tau \le C^T \int_{t_0}^{t_1} [\int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s)] u_1(\tau) d\tau$$

$$= \int_{t_0}^{t_1} \left| C^T \int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s)] u(\tau) \right| d\tau \quad (2.10)$$

and

$$U(t) = \operatorname{sgn}[\int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s)] u(\tau) \quad (2.11)$$

As y_1 is on the boundary. Given that we've always $0 \in R(t_0, t_1)$; If zero weren't located inside $R(t_0, t_1)$, then it is on the boundary. Therefore, based on the previous logic, this suggests that

$$0 = \int_{t_0}^{t_1} \left| C^T [\int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s)] u(\tau) \right| d\tau \quad (2.12)$$

So that $C^T \int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s) = 0$ at most everywhere on $t \in [t_0, t_1]$

By its own definition, this indicates that a system is improper, since $c \neq 0$.

Hence, if $0 \in \operatorname{int} \; R(t_0, t_1)$, then

$$C^T \int_{-h}^{0} X(t_1, \tau - s) d\overline{H}(\tau - s, s) = 0$$

at most everywhere on $t \in [t_0, t_1]$ would imply $c = 0$ proving properness of system (1.1), We determine the system's relative controllability (1.1) for every interval.

3.1OPTIMALITY CONDITION OF THE SYSTEM

Now, we go back to our initial objective of striking a constantly moving target $z(t)$ in least time. Consider the trajectory of the system (1.1) given by

$$x(t_1, t_0, \phi, u) = X(t_1, t_0, \phi(0)) + [\int_{-h}^{0} ds \overline{H}(\tau - s, s) u(\tau, s) d\tau$$

Or equivalently $W(t) = Z(t) - x(t)$ (3.1)

Then reaching $Z(t)$ at time t corresponds to $Z(t) - x(t) \cong W(t) \in R(t, 0)$

We now show that if u^* is the optimal control with time t^* the optimal time, then

$$Z(t^*) = X(t^*, 0)\phi(0) - \int_{-h}^{0} X(t^*, \tau - s)[\int_{-h}^{0} ds \overline{H}(t^*, \tau) u_0 ds \equiv u(t^\tau) t \partial R(t^*, 0) \quad (3.2)$$

That is $u(t^*)$ is on the border of the reachable set that is limited.

Theorem 3.1
Let $u^*(t)$ be the optimal control with t^* the minimum time, then $u(t^*) \in \partial R(t^*, 0)$, the boundary of $R(t^*, 0)$

Proof:
Assume u^* is used to hit $\omega(t)$ in time t^*, then

$$Z(t^*)-X(t^*,0)\phi(0)-\int_{t_0}^{t_1}X(t^*,\tau-s)[\int_{-h}^{0}ds\overline{H}(t^*,\tau)u_0d\tau \equiv U(t^*)\in R(t^*,0) \quad (3.3)$$

Assume $u(t^*)$ is not on the boundary then $u(t^*)\in \text{int}\ R(t^*,0),\ t^*>0$. Hence there exists a ball $B(u(t^*),r)$ of radius r, about $u(t^*)$ such that $B(u(t^*),r)\in R(t^*,0)$ because $R(t^*,0)$ is a continuous function of t, there exists a $d>0$ such that $B(u(t^*),r)\in R(t^*,0)$ for $t^*-\delta\le t\le t^*$. Therefore $u(t^*)\in\partial R(t^*,0)$ for $t-\delta\le t$. This contradicts the optimality of t^*, hence $u(t^*)\in\partial R(t^*,0)$

Theorem 3.2

If u^* be an optimal control transferring system (1.1) from $x(0)$ to $Z(t^*)$ in minimum time, t^*, then there exists a non-zero function $C\in E^n$ such that

$$u^*(t)=\text{sgn}\{C^TX(t,\tau-s)\overline{H}\} \quad (3.4)$$

Proof:

Define $y(t)-X(t,\tau-s)\overline{H}$

$$u(t^*)=Z(t^*)-X(t^*,0)\phi(0)-\int_{t_0}^{t_1}X(t^*,\tau-s)[\int_{-h}^{0}ds\overline{H}(t^*,\tau)u_0d\tau \quad (3.5)$$

That is, $u(t^*)=\int_{t_0}^{t_1}X(t^*,\tau-s)Hu^*(\tau)d\tau$

$$H=\int_{t_0}^{t^*}X(t^*,\tau-s)Hu^*(\tau)d\tau$$

From theorem (3.1), $u(t^*)$ is on the boundary $\partial R(t^*,0)$ of constrained reachable set. The supporting hyper plane theorem (see Hermes and Lasalle[4]) then implies the existence of a non trivial hyperplane with outward normal c (say) supporting $\partial R(t^*,0)$ at $u(t^*)$. In otherwords,

$C^Tu(t^*)\ge c^Ty$ for all $y\in R(t^*,0)$

That is

$$C^T\int_{t_0}^{t_1}X(t^*,\tau-s)Hu^*(\tau)d\tau\ge C^T\int_{t_0}^{t^*}X(t^*,\tau-s)Hu(\tau)d\tau$$

for all $u\notin U$

Rearranging gives

$$C^T\int_{t_0}^{t^*}X(t^*,s)H[u^*(\tau)-u(\tau)]d\tau\ge 0$$

This can happen only if

$$u^*=\text{sgn}\{C^TX(t^*,\tau-s)H\} \quad (3.6)$$

CONCLUSION

Consider the Simple Harmonic Oscillator given by

$$\ddot{x}+x=u(t);\quad |u|\le 1$$

The principal matrix solution of (4.1) above is

$$X(t)=\begin{pmatrix} Cost & S\text{int} \\ -S\text{int} & Cost\end{pmatrix}$$

From which we infer that

$$X^{-1}(t)=\begin{pmatrix} Cost & -S\text{int} \\ S\text{int} & Cost\end{pmatrix}$$

We can easily verify by the Kalman Rank condition that the system is controllable, that is rank $(B,AB)=n=2$

Also, the eigenvalues are $\pm i$ indicating non – negative real parts. Hence, by Brunday [10], the solution is uniformly asymptotically stable. This solution goes to zero as $t\to\infty$. Given that the system can be controlled, there exists an optimal control $u^*(t)$ that drives the solution to the origin in finite time t. this optimal control is of the form of Hermes and Lasalle in [4]

That is $\text{sgn}(C^TY(t))$, where

$$C^TY(t)=\begin{pmatrix}C_1 & C_2\end{pmatrix}\begin{pmatrix} Cost & S\text{int} \\ -S\text{int} & Cost\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$=C_1S\text{int}+C_2Cost\equiv(C_1^2+C_2^2)(Sin(t+\delta));t\le\delta\le\pi$$

$$\operatorname{sgn}(C^T Y(t) = \operatorname{sgn}[\sin(t+\delta)] = \begin{cases} 1 \;\; if\; Sin(t+\delta) > 0 \\ 0 \;\; if\; Sin(t+\delta) = 0 \\ -1 \;\; if\; Sin(t+\delta) < 0 \end{cases} \qquad (4.5)$$

This illustrate that the Simple Harmonic Oscillator can optimally be controlled.

CONCLUSION

In conclusion, we have shown that a linear delay system with distributed delays in state and control can be relatively controlled if the system is proper and the controllability grammian is non-singular. We also show that a necessary condition for existence of the optimal control is that it must be on the boundary of the reachable set. We proceeded by showing the form of the optimal control for the system in question. Finally, we join Chukwu in [3] and Onwuata in [6] can get the conclusion that optimal control is distinct and BangBang if a system is reasonably controllable.

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