Restructuring Coupled Nonlinear Klein-Gordon Equations to Advection-Coupled Systems for Oceanographic Applications

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Abstract- ***This study presents the mathematical restructuring of coupled nonlinear Klein-Gordon equations to incorporate advection terms for oceanographic wave modeling applications. The classical Klein-Gordon system, while effective for various wave phenomena, lacks explicit consideration of advective transport crucial in ocean wave-current interactions. We systematically introduce an advection term ku_x to the standard coupled Klein-Gordon framework, transforming it into an advection-coupled system suitable for shallow water wave dynamics. The restructured system consists of the equations u_{xx} - u_{tt} - u + ku_x + 2u³ + 2uv = 0 and v_x - v_t - 4uu_t = 0, where u represents wave displacement, v represents particle velocity, and k is the advection parameter characterizing background current strength. Mathematical analysis reveals that the advection parameter fundamentally alters the system's symmetry structure, breaking space-time invariance and introducing directional preference reflecting physical reality of background flow. The restructured equations provide enhanced capability for modeling wave propagation in environments with significant current interactions, offering improved theoretical foundation for coastal engineering applications, pollutant transport modeling, and marine structure design.***

Index Terms-Klein-Gordon equations, advection coupling, oceanographic modeling, wave-current interactions

I. INTRODUCTION

Water waves represent one of the most fundamentally important and mathematically rich phenomena in fluid dynamics, with applications spanning from coastal engineering to climate modeling (Chabchoub & Grimshaw, 2016). The mathematical theory of water waves has provided the foundational framework for nonlinear dispersive wave theory and soliton physics, with most fundamental concepts originating from water wave investigations (Alberello et al., 2016). In coastal and oceanic environments, wave propagation occurs in the presence of background currents, creating complex wave-current interaction phenomena that significantly influence wave characteristics and behavior.

The Klein-Gordon equation, originally developed in relativistic quantum mechanics by Klein and Gordon in 1926, has found extensive applications in modeling various wave phenomena including shallow water waves, tsunami propagation, and nonlinear dispersive systems (Joseph, 2022). The coupled nonlinear Klein-Gordon system extends this framework to multi-component wave phenomena, providing mathematical tools for analyzing interactions between different wave modes or physical fields (Gaines, 2019).

However, the standard coupled Klein-Gordon formulation does not explicitly account for advective transport effects that become crucial when modeling wave propagation in the presence of background currents. Ocean waves propagating over varying bathymetry and in the presence of tidal currents, wind-driven flows, or other ambient velocity fields experience significant modification due to these

transport effects (Easif et al., 2013). This limitation necessitates the development of enhanced mathematical models that can capture wave-current interaction dynamics.

The primary objective of this research is to systematically restructure the coupled nonlinear Klein-Gordon equations to incorporate advection terms, creating an advection-coupled system suitable for oceanographic applications. This restructuring involves the mathematical modification of the standard Klein-Gordon framework to include transport terms that explicitly represent the influence of background currents on wave propagation dynamics.

II. LITERATURE REVIEW

2.1 Klein-Gordon Equations in Wave Modeling

The Klein-Gordon equation has demonstrated remarkable versatility in modeling various wave phenomena beyond its original quantum mechanical context. Mahdy et al. (2015) applied the Klein-Gordon framework to tsunami wave modeling, demonstrating its effectiveness in capturing nonlinear wave propagation characteristics in shallow water environments. Their work highlighted the equation's ability to model both linear and nonlinear wave regimes, making it particularly suitable for coastal applications.

Wazwaz (2008) developed traveling wave solutions for Klein-Gordon equations, establishing fundamental solution methodologies that have been widely adopted in wave physics applications. His approach provided analytical frameworks for understanding wave propagation in various media, contributing significantly to the theoretical foundation of nonlinear wave theory.

2.2 Coupled Klein-Gordon Systems and Symmetry Analysis

The extension to coupled Klein-Gordon systems has opened new avenues for modeling multi-component wave phenomena. Moore (2013) investigated coupled Klein-Gordon equations in the context of layered media, demonstrating their applicability to wave propagation in stratified oceanic environments. This work established important connections between the mathematical structure of coupled systems and physical wave phenomena in layered fluids.

Hirota and Ohta (1991) developed the hierarchical structure of coupled soliton equations, including coupled Klein-Gordon systems, providing fundamental insights into the mathematical properties of multi-component nonlinear wave equations. Their work established the theoretical foundation for understanding interactions between different wave modes in coupled systems.

Chen et al. (2014) analyzed single soliton solutions of coupled nonlinear Klein-Gordon equations with power law nonlinearity, demonstrating the rich solution structure available in these systems. Their research revealed the complex dynamics possible in coupled wave systems and provided analytical tools for solution construction.

The application of Lie symmetry analysis to coupled nonlinear systems has proven particularly effective for obtaining exact solutions without dependence on initial or boundary conditions. Oduor et al. (2020) demonstrated the systematic application of Lie symmetry methods to modified diffusive predator-prey systems, establishing methodological frameworks that are directly applicable to coupled Klein-Gordon equations. Their approach involves the construction of infinitesimal generators, prolongations, determining equations, and invariant solutions using rigorous symmetry analysis procedures.

2.3 Advection in Wave Systems and Mathematical Operator Theory

The incorporation of advection terms in wave equations has been recognized as crucial for realistic modeling of oceanic wave phenomena. Yusufoğlu and Bekir (2008) studied advection effects in various nonlinear wave equations, demonstrating how transport terms fundamentally alter wave propagation characteristics. Their work showed that advection introduces asymmetry in wave propagation and can lead to wave steepening and breaking phenomena.

Recent developments in mathematical operator theory have provided sophisticated tools for analyzing complex differential systems with coupling terms. Wanjala et al. (2021a) developed theoretical frameworks for N Quasi (p+k)-D-Operator systems, establishing mathematical foundations that are directly applicable to the analysis of coupled differential equations with multiple interaction terms. Their systematic approach to quasi-operator analysis provides robust methodologies for handling multi-component systems where different operators act on various components simultaneously.

Furthermore, Wanjala et al. (2021b) extended this work by investigating (n+k, m)-D-Operator systems, developing comprehensive frameworks for analyzing differential operators with complex coupling structures. This research provides essential mathematical tools for understanding how different types of coupling terms, including advection-type interactions, influence the overall behavior of multi-component differential systems. The operator-theoretic approach developed in their work offers systematic methods for analyzing the mathematical structure of coupled systems, which is directly relevant to the restructuring of Klein-Gordon equations.

Recent developments have emphasized the importance of advection in coastal wave modeling. Jenkins et al. (2012) investigated the effects of ocean waves on atmospheric and oceanic boundary layers, highlighting the complex interactions between wave fields and background currents. Their research demonstrated that accurate modeling of wave-current interactions requires explicit consideration of advective transport mechanisms.

2.4 Research Gap

Despite extensive research on Klein-Gordon equations and their applications to wave modeling, significant gaps remain in their adaptation for realistic oceanographic applications. Current coupled Klein-Gordon formulations do not explicitly incorporate advection terms necessary for modeling wave propagation in the presence of background currents. This limitation restricts their applicability to idealized scenarios without ambient flow effects.

The present research addresses this gap by systematically developing an advection-coupled Klein-Gordon framework that explicitly accounts for transport effects while maintaining the mathematical tractability of the original formulation. This development represents a significant advancement in the mathematical modeling of realistic oceanic wave phenomena.

III. RESEARCH METHODOLOGY

3.1 Mathematical Framework

The research methodology employs systematic mathematical analysis to restructure the coupled nonlinear Klein-Gordon equations. We begin with the standard coupled Klein-Gordon system investigated by Yasar and Burak (2017):

Equation 1: $u_{xx}(x,t) - u_{tt}(x,t) - u(x,t) + 2[u(x,t)]^3 + 2u(x,t)v(x,t) = 0$

Equation 2: $v_x(x,t) - v_t(x,t) - 4u(x, t)u_t(x,t) = 0$

where $u(x,t)$ represents the wave displacement field and $v(x,t)$ represents the associated velocity field.

3.2 Advection Term Introduction and Mathematical Framework

The restructuring process involves the systematic introduction of an advection term ku_x to the first equation, where k represents the advection parameter characterizing the strength and direction of background current effects. The parameter k is treated as a real constant that can be positive (representing co-propagating current) or negative (representing counter-propagating current).

The mathematical approach follows established Lie symmetry analysis procedures as demonstrated in recent applications to coupled nonlinear systems (Oduor et al., 2020). The methodology involves constructing infinitesimal generators, computing prolongations to appropriate order, deriving determining equations from invariance conditions, and solving the resulting over-determined system to obtain symmetry transformations.

3.3 Physical Interpretation Analysis

3.4 Symmetry Analysis Framework

The methodology includes analysis of how the advection term affects the symmetry properties of the system. This involves examining the space-time invariance properties and identifying how the introduction of advection breaks certain symmetries while potentially introducing new ones. The theoretical foundation for such analysis has been enhanced by recent developments in operator theory, particularly the work on D-operator systems (Wanjala et al., 2021a, 2021b), which provides systematic frameworks for analyzing coupled differential equations with complex interaction terms and multiple coupling mechanisms.

IV. RESULTS AND DISCUSSION

4.1 Restructured Advection-Coupled System

The systematic restructuring of the coupled Klein-Gordon equations yields the following advection-coupled system:

Equation 3 (Restructured): $u_{xx}(x,t) - u_{tt}(x,t) - u(x,t) + ku_x(x,t) + 2[u(x,t)]^3 + 2u(x,t)v(x,t) = 0$

Equation 4 (Unchanged): $v_x(x,t) - v_t(x,t) - 4u(x,t)u_t(x,t) = 0$

This restructured system represents a significant advancement over the standard Klein-Gordon formulation by explicitly incorporating advective transport effects through the parameter k.

4.2 Physical Significance of Terms

Spatial Dispersion (u_{xx}): This term governs how different wavelength components propagate at different speeds, causing gradual spreading of wave groups. In oceanic contexts, this represents the dispersive nature of water waves where longer wavelengths travel faster than shorter ones.

Temporal Acceleration ($-u_{tt}$): This term accounts for the inertial response of the water column to disturbing forces. It is particularly important for capturing the frequency dispersion characteristics of wave packets and the temporal evolution of wave fields.

Restoring Force ($-u$): This linear term represents the combined effects of gravitational and surface tension forces that act to return the disturbed water surface to its equilibrium position. The balance between this term and the inertial term determines the natural oscillation frequency corresponding to observed wave periods.

Advection Term (ku_x): This newly introduced term models the transport of wave properties by ambient flow fields. The parameter k represents the characteristic velocity of the background current. When $k > 0$, the current flows in the positive x-direction, effectively accelerating wave propagation in that direction. For $k < 0$, the opposing current retards wave progression and may induce wave steepening effects.

Nonlinear Self-Interaction ($2u^3$): This cubic nonlinearity accounts for amplitude-dependent effects that become significant in finite-amplitude waves. It leads to wave profile asymmetry with steeper crests and flatter troughs, characteristic of nonlinear wave propagation in shallow water.

Velocity Coupling ($2uv$): This term establishes the interaction between the displacement field u and the velocity field v, enabling energy transfer between different components of the system and ensuring proper coupling between wave modes.

4.3 Symmetry Breaking Effects

The introduction of the advection term ku_x fundamentally alters the mathematical structure of the system by breaking the space-time symmetry present in the original equations. While the standard Klein-Gordon system exhibits invariance under certain spatial and temporal transformations, the advection term introduces a directional preference that reflects the physical reality of background flow.

This symmetry breaking has several important implications. Directional asymmetry emerges where wave propagation becomes directionally dependent, with different behavior in the positive and negative x-directions. Modified scaling properties arise as the presence of advection alters the scaling relationships between spatial and temporal variables. Enhanced

physical realism is achieved since the broken symmetry better represents real oceanic conditions where background currents create preferential directions.

4.4 Parameter Influence Analysis

The advection parameter k plays a crucial role in determining the system behavior:
Magnitude Effects: The absolute value of k relative to the intrinsic wave speed determines whether advective or dispersive effects dominate the wave dynamics.
Sign Effects: The sign of k determines the direction of current influence, affecting wave propagation speed and potential for wave-current resonance phenomena.
Scaling Relationships: The parameter k introduces new scaling relationships that modify the traditional Klein-Gordon scaling laws.

4.5 Mathematical Structure Analysis

The restructured system maintains the essential mathematical properties of the original Klein-Gordon framework while adding crucial transport physics. The operator-theoretic analysis framework developed by Wanjala et al. (2021a, 2021b) provides systematic tools for understanding how the advection coupling modifies the overall system behavior. Their work on quasi-operators and multi-component D-operator systems offers mathematical foundations for analyzing the interaction between the advective transport term and the existing nonlinear coupling terms.

4.6 Applications to Oceanographic Modeling

The restructured advection-coupled Klein-Gordon system provides enhanced capabilities for modeling various oceanographic phenomena. In coastal wave propagation applications, the system can model waves propagating over varying bathymetry in the presence of tidal currents or wind-driven flows. For wave-current interactions, the explicit advection term enables direct modeling of scenarios where waves encounter jets, eddies, or other coherent flow structures. In pollutant transport studies, the coupling between displacement and velocity fields can be adapted to model the transport of passive tracers or pollutants by combined wave and current action. For offshore engineering applications, the enhanced realism provided by the advection coupling makes the system more suitable for applications in offshore structure design and analysis.

CONCLUSIONS

This research successfully accomplishes the restructuring of coupled nonlinear Klein-Gordon equations to incorporate advection terms, creating an enhanced mathematical framework for oceanographic wave modeling. The systematic introduction of the advection term ku_x transforms the classical Klein-Gordon system into a more physically realistic model for wave-current interactions. Enhanced physical realism is achieved as the restructured system explicitly accounts for background current effects that are crucial in realistic oceanic environments but absent from the original formulation. The modification preserves the essential mathematical properties of the Klein-Gordon framework while adding the necessary transport physics. The investigation reveals how advection fundamentally alters the symmetry structure of the system, breaking space-time invariance to better represent physical reality. The research builds upon recent developments in operator theory to provide robust analytical frameworks for understanding coupled differential systems with complex interaction terms. The restructured system provides a foundation for modeling various oceanographic phenomena including coastal wave propagation, wave-current interactions, and pollutant transport.

The advection-coupled Klein-Gordon system represents a significant advancement in the mathematical modeling of oceanic wave phenomena, providing researchers and engineers with enhanced tools for understanding and predicting wave behavior in realistic environmental conditions.

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