Generalized Lindley Probability Distribution Up to Five Parameters

DR MALAY SANYAL[1], SAYANTANI GHOSH[2]

[1,2]*Department of Mathematics, Ranaghat College, University of Kalyani, Ranaghat 741201, West Bengal, India*

Abstract- In this paper, we extend the Alpha Power Transformed Lindley (APT Lindley) distribution model by introducing an additional parameter. This new model represents a distribution that significantly enhances flexibility to express real-life phenomena. The probability density function, cumulative distribution function, hazard function and survival function are derived and discussed. We also explore characteristics such as moments, moment-generating functions and quantile functions. The maximum likelihood estimation method is employed for parameter estimation of the model. This newly proposed distribution is well-suited for real-world (non-fictional) datasets and reveals interesting properties due to the flexible nature of its hazard function. It presents a promising alternative in statistical and probability theory particularly for applications in health, engineering and economics.

Index Terms- Alpha Power Transformed, Lindley distribution, Maximum Likelihood estimators, quantile function

I. INTRODUCTION

Probability distributions are highly valuable in real-world scenarios for modelling and evaluating occurrences. Numerous classical distributions are utilised across a variety of sectors. Appropriate probability distributions are used to model financial variables (such as stock prices, interest rates and asset returns) and the lifetime characteristics of systems. This distribution recognizes a real-life problem by allocating a range of possible outcomes to the variables. Recent developments have focused on defining new families of distributions that extend familiar ones while providing greater flexibility in modeling data.

Lindley [1] introduced the Lindley distribution to analyze failure time data. This distribution is primarily applied to model stress-strength reliability. The motivation for the Lindley distribution stems from its ability to stimulate various hazard rates including monotonically increasing, decreasing, constant, bathtub and increasing-decreasing-increasing forms.

In Bayesian statistics, the Lindley distribution was initially introduced by Lindley [1]. A random variable X>0 is said to follow the Lindley distribution with scale parameter θ>0 if its probability density function (pdf) is given by f(x;θ)=(θ^2 (1+x) e^(-θx))/(θ+1),as per [6,8]. This density function is created as a mixture of the two distributions : The exponential distribution Exp(θ) and the Gamma distribution Gamma(2, θ).

Many researchers aim to generalize the Lindley distribution because its increasing failure rate may not always comfortably fit real-life data. Consequently, several classes of distribution have been introduced by adding one or more parameters. Bakouch et al.[4] proposed a distribution offering a better alternative to the exponential failure time distributions, which often lack unimodal and bathtub-shaped failure rates. Ghitany et al. [8,14] suggested that the Lindley distribution is superior to the exponent distribution when the hazard rate is unimodal and bathtub-shaped. J.Mazucheli et al.[16] proposed the Lindley distribution as a variable substitute for exponential and Weibull distributions.

The power Lindley (PL) distribution was proposed by Ghitany et al. [2], and the two-parameter Lindley (TPL) distribution was introduced by Shanker et al. [3]. These distributions extend the original Lindley distribution proposed by [1]. Methods for generating new distributions were proposed by Nadaraj et al. [5] and additional methods can be found in [24,29,30] ; further information is available in [31,11].

A. Another important method for generating new distribution with notable flexibility in modelling real-life data is alpha power transformation (APT), proposed by Mahdavi and Kundu [7]. This method, involving the parameter α, generates univariates distributions capable of modelling both monotone and non-monotone failure rate functions. It incorporates skewness into the base distribution using α. the cumulative distribution function (*cdf*) and probability density function (*pdf*) of a random variable under ATP are specified as follows

$$F(x;\alpha) = \frac{\alpha^{G(x)}-1}{\alpha-1}\ , if\ \alpha > 0, \alpha \neq 1,$$

$$=G(x)\ ,\quad if\ \alpha = 1, \qquad (1)$$

$$f(x;\alpha) = \frac{\log(\alpha) g(x)\ \alpha^{G(x)}}{\alpha-1}\ , if\ \alpha > 0, \alpha \neq 1,$$

$$=g(x)\ ,\qquad if\ \alpha = 1, \quad (2)$$

$$g(x;\theta,\delta,\beta) = \frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x} \qquad (3)$$

$$G(x;\theta,\delta,\beta) = 1-(1+\frac{\theta\beta x}{\theta\delta+\beta})e^{-\theta x} \qquad (4)$$

Where $x > 0, \theta > 0, \beta > 0, \delta > -\frac{\beta}{\theta}$

The hazard function h (x;β,θ,δ) of the three-parameter Lindley probability distribution is always monotonically increasing, according to Shanker et al. [3]. This monotonic property is not always suitable for modelling real-life data. Thus, researchers seek to extend the distribution to create a more flexible hazard function. Generalizations of existing distribution are necessary to improve model flexibility.

Using the APT model of Mahdavi and Kundu [7] from equations (1) and (2), S.J.Dugasa et al.[13] derived a new distribution from the base Lindley distributed random variable X with pdf g(x) and cdf G(x) are given in equations (3) and (4). According to S.J.Dugasa et al.[13], a random variable X forms an APT Lindley probability distribution. Its cdf and pdf with transformation parameter α>0,α≠1 are given by

$$F_{APTLD}(x;\beta,\theta,\delta,\alpha) = \frac{\alpha^{1-(1+\frac{\theta\beta x}{\theta\delta+\beta})e^{-\theta x}}-1}{\alpha-1}$$

$$f_{APTLD}(x;\beta,\theta,\delta,\alpha) = \frac{\log(\alpha)}{(\alpha-1)}\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x}\ \alpha^{1-(1+\frac{\theta\beta x}{\theta\delta+\beta})e^{-\theta x}}$$

This paper aims to extend the APT Lindley distribution model by introducing a new parameter ϵ>0,resulting in a new distribution with improved flexibility for modelling real-life data. The APT Lindley distribution model, by incorporating the parameters α and ϵ adjusts for skewness in the base distribution, making it suitable for modelling skewed data that may not fit standard distribution. The model has practical applications in biomedical research, industrial reliability and engineering. This work may serve as a foundation for future engineering, education and economics.

II. MATHEMATICAL FORMULATION

From the approach of Mahdavi and Kundu [7] expressed in equations (1) and (2), we construct a new distribution from the base Lindley distribution random variable X whose *pdf* $g(x)$ and *cdf* $G(x)$ are given in equation (3) and (4). Now we proposed five parameters probability density function and corresponding probability distribution function as the extended version of Lindley probability distribution.

$$f(x;\theta,\delta,\beta,\alpha,\epsilon) = \frac{(\epsilon-1+\log\alpha)e^{(\epsilon-1)G(x)}\alpha^{G(x)}g(x)}{(\alpha e^{\epsilon-1}-1)}\quad ;\epsilon,\alpha > 0\ \&\ \epsilon \neq \alpha \neq 1 \qquad (5)$$

(newly proposed five parameters probability density function of Lindley Distribution)

$$= \frac{\log\alpha\ \alpha^{G(x)}g(x)}{\alpha-1}\epsilon = 1, \alpha \neq 1$$

(6)

(probability density function of Alpha Power Transformed Lindley Distribution)

$$= \frac{(\epsilon-1)g(x)e^{(\epsilon-1)G(x)}}{e^{\epsilon-1}-1} \; ; \epsilon \neq 1, \alpha = 1 \quad (7)$$

(newly proposed five parameters probability density function of Lindley Distribution)

$$= g(x) \; ; \; \epsilon = \alpha = 1 \quad (8)$$

(probability density function of Lindley Distribution with three parameters)

$$F(x;\theta,\delta,\beta,\alpha,\epsilon) = \frac{e^{(\epsilon-1)G(x)}\alpha^{G(x)}-1}{(\alpha e^{\epsilon-1}-1)} \; ; \epsilon, \alpha > 0$$

$$\& \; \epsilon \neq \alpha \neq 1 \quad (9)$$

(newly proposed five parameters probability distribution function of Lindley Distribution)

$$= \frac{\alpha^{G(x)}-1}{\alpha-1} \; ; \epsilon = 1, \alpha \neq 1 \quad (10)$$

(probability distribution function of Alpha Power Transformed Lindley Distribution)

$$= \frac{e^{(\epsilon-1)G(x)}-1}{e^{\epsilon-1}-1} \; ; \epsilon \neq 1, \alpha = 1 \quad (11)$$

(newly proposed five parameters probability distribution function of Lindley Distribution)

$$= G(x) \; ; \; \epsilon = \alpha = 1 \quad (12)$$

(probability distribution function of Lindley Distribution with three parameters)

Theorem 2.1. Let X be a random variable having five parameters Lindley probability distribution. Then, its cdf and pdf with transformation parameter

$\boldsymbol{\epsilon, \alpha > 0}$ & $\boldsymbol{\epsilon \neq \alpha \neq 1}$ respectively

$$F(x;\theta,\delta,\beta,\alpha,\epsilon) = \frac{e^{(\epsilon-1)G(x)}\alpha^{G(x)}-1}{(\alpha e^{\epsilon-1}-1)}$$

$$f(x;\theta,\delta,\beta,\alpha,\epsilon) = \frac{(\epsilon-1+log\alpha)e^{(\epsilon-1)G(x)}\alpha^{G(x)}g(x)}{(\alpha e^{\epsilon-1}-1)}$$

[For the case of $\epsilon = \alpha = 1$, the new cdf and pdf assume the base Lindley distribution.]

Proof.

(a) (a) For *cdf,* $F(x;\theta,\delta,\beta,\alpha,\epsilon) = \frac{e^{(\epsilon-1)G(x)}\alpha^{G(x)}-1}{(\alpha e^{\epsilon-1}-1)}$

$$= \frac{e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\alpha^{[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}-1}{(\alpha e^{\epsilon-1}-1)}$$

For *pdf,* $f(x;\theta,\delta,\beta,\alpha,\epsilon) = \frac{(\epsilon-1+log\alpha)e^{(\epsilon-1)G(x)}\alpha^{G(x)}g(x)}{(\alpha e^{\epsilon-1}-1)}$

$$= \frac{(\epsilon-1+log\alpha)e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\alpha^{[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x}}{(\alpha e^{\epsilon-1}-1)}$$

To show integration of *pdf* is unity,

$$\int_0^\infty f(x;\theta,\delta,\beta,\alpha,\epsilon)$$

$$= \int_0^\infty \frac{(\epsilon-1+log\alpha)e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\alpha^{[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x}}{(\alpha e^{\epsilon-1}-1)}$$

$$let, u = 1 - (1 + \frac{\theta\beta x}{\theta\delta+\beta})e^{-\theta x}$$

$$dx = \frac{(\theta\delta+\beta)du}{\theta^2(\delta+\beta x)e^{-\theta x}}$$

$$I = \int e^{(\epsilon-1)u}\alpha^u du \quad = \frac{e^{(\epsilon-1)u}\alpha^u}{\epsilon-1+log\alpha}$$

$$\int_0^\infty f(x;\theta,\delta,\beta,\alpha,\epsilon)$$

$$= \int_0^\infty \frac{(\epsilon-1+log\alpha)e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\alpha^{[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x}}{(\alpha e^{\epsilon-1}-1)}$$

$$= \frac{(\epsilon-1+log\alpha)}{(\alpha e^{\epsilon-1}-1)}\int_0^1 e^{(\epsilon-1)u}\alpha^u =$$

$$\frac{(\epsilon-1+log\alpha)}{(\alpha e^{\epsilon-1}-1)}\frac{[e^{(\epsilon-1)u}\alpha^u]_0^1}{(\epsilon-1+log\alpha)} = 1$$

Theorem 2.2. Let X be a random variable having four parameters Lindley probability distribution can be obtained by simply putting α=1 ,where ϵ≠1. Then, its cdf and pdf with transformation parameter

$$F(x;\theta,\delta,\beta,\epsilon) = \frac{e^{(\epsilon-1)G(x)}-1}{e^{\epsilon-1}-1}$$

$$f(x;\theta,\delta,\beta,\epsilon)=\frac{(\epsilon-1)g(x)e^{(\epsilon-1)G(x)}}{e^{\epsilon-1}-1}$$

[For the case of $\epsilon=\alpha=1$, the new cdf and pdf assume the base Lindley distribution.]

Proof.

(a) For *cdf* , $F(x;\theta,\delta,\beta,\epsilon)=\frac{e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}-1}{e^{\epsilon-1}-1}$

(b) For *pdf* , $f(x;\theta,\delta,\beta,\epsilon)$ =

$$\frac{(\epsilon-1)\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x}e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}}{e^{\epsilon-1}-1}$$

Similarly, we obtain integration of *pdf* is unity

$$\int_0^\infty f(x;\theta,\delta,\beta,\epsilon)=\int_0^\infty\frac{(\epsilon-1)\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x}e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}}{e^{\epsilon-1}-1}=1$$

III. SURVIVAL AND HAZARD FUNCTIONS OF THE NEWLY SUGGESTED FIVE PARAMETERS LINDLEY DISTRIBUTION

The survival and Hazard function of the newly suggested five parameters Lindley distribution can be derived based on their definitions.

3.1. The survival and Hazard function of the newly suggested five parameters Lindley distribution with parameters $\boldsymbol{\epsilon,\alpha>0}$ & $\boldsymbol{\epsilon\neq\alpha\neq1}$ *can be obtained as follows:*

The survival function is,

$$S(x;\theta,\delta,\beta,\alpha,\epsilon)=1-F(x;\theta,\delta,\beta,\alpha,\epsilon)=\frac{e^{\epsilon-1}\alpha[1-e^{(\epsilon-1)\left[-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}\alpha^{[-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}]}{\alpha e^{\epsilon-1}-1}$$

(13)

The hazard function is,

$$h(x;\theta,\delta,\beta,\alpha,\epsilon)=\frac{f(x;\theta,\delta,\beta,\alpha,\epsilon)}{S(x;\theta,\delta,\beta,\alpha,\epsilon)}$$

$$=\frac{(\epsilon-1+\log\alpha)e^{(\epsilon-1)[-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\alpha^{[-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\theta^2(\delta+\beta x)e^{-\theta x}}{[1-e^{(\epsilon-1)\left[-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}\alpha^{\left[-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}](\theta\delta+\beta)}$$

(14)

3.2. The survival and Hazard function of the newly suggested four parameters Lindley distribution can be obtained by simply putting $\boldsymbol{\alpha=1}$ *,where* $\boldsymbol{\epsilon\neq1}$.

Thus, the survival function is

$$S(x;\theta,\delta,\beta,\epsilon)=1-F(x;\theta,\delta,\beta,\epsilon)=1-\frac{e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}-1}{e^{\epsilon-1}-1}$$

$$=\frac{e^{\epsilon-1}[1-e^{-(\epsilon-1)\left[\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}]}{e^{\epsilon-1}-1}$$

(15)

The hazard function is

$$h(x;\theta,\delta,\beta,\epsilon)=\frac{f(x;\theta,\delta,\beta,\epsilon)}{S(x;\theta,\delta,\beta,\epsilon)}=$$

$$=\frac{(\epsilon-1)\theta^2(\delta+\beta x)e^{-\theta x}}{[e^{\epsilon-1}-e^{(\epsilon-1)\left[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}e^{(\epsilon-1)\left[-1+\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}](\theta\delta+\beta)}$$

(16)

IV. MOMENTS AND MOMENT-GENERATING FUNCTION

4.1. The moments and moments-generating function of the newly suggested five parameters Lindley distribution with parameters $\boldsymbol{\epsilon,\alpha>0}$ & $\boldsymbol{\epsilon\neq\alpha\neq1}$ *can be derived as follows:*

The r^th moment of a random variable is used to determine the mean, variance, skewness and kurtosis and is generally defined as:

$$\mu_r^i=E(X^r)=\int_0^\infty X^r f(x;\theta,\delta,\beta,\alpha,\epsilon)dx$$

$$=\int_0^\infty X^r\frac{(\epsilon-1+\log\alpha)e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\alpha^{[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\theta^2(\delta+\beta x)e^{-\theta x}}{(\alpha e^{\epsilon-1}-1)(\theta\delta+\beta)}dx$$

$$= \frac{(\epsilon-1+log\alpha)\theta^2\alpha e^{\epsilon-1}}{(\theta\delta+\beta)(\alpha e^{\epsilon-1}-1)}\int_0^\infty X^r e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}\alpha^{[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}(\delta+\beta x)e^{-\theta x}dx \quad (17)$$

We can apply,

$$e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]} = \sum_{m=0}^{\infty}\frac{(-1)^m(\epsilon-1)^m\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^m e^{-\theta m x}}{m!},$$

$$\alpha^{[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]} = \sum_{n=0}^{\infty}\frac{(-1)^n(log\alpha)^n\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^n e^{-\theta n x}}{n!}$$

Now from (17) we get,

$$\mu_r^i = \frac{(\epsilon-1+log\alpha)\theta^2\alpha e^{\epsilon-1}}{(\theta\delta+\beta)(\alpha e^{\epsilon-1}-1)}\int_0^\infty X^r(\delta+\beta x)e^{-\theta x}\sum_{m=0}^{\infty}\frac{(-1)^m(\epsilon-1)^m\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^m e^{-\theta m x}}{m!}\sum_{n=0}^{\infty}\frac{(-1)^n(log\alpha)^n\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^n e^{-\theta n x}}{n!}dx$$

$$= \frac{(\epsilon-1+log\alpha)\theta^2\alpha e^{\epsilon-1}}{(\theta\delta+\beta)(\alpha e^{\epsilon-1}-1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(\epsilon-1)^m(log\alpha)^n}{m!\,n!}\int_0^\infty X^r(\delta+\beta x)e^{-\theta(1+m+n)x}\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^{m+n}dx$$

$$= \frac{(\epsilon-1+log\alpha)\alpha e^{\epsilon-1}}{(\alpha e^{\epsilon-1}-1)}\sum_{i=0}^{m+n}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(\epsilon-1)^m(log\alpha)^n}{m!\,n!}\frac{\beta^i\theta^{i+2}}{(\theta\delta+\beta)^{i+1}}\binom{m+n}{i}\int_0^\infty x^{r+i}(\delta+\beta x)e^{-\theta(1+m+n)x}dx$$

[By using series expansion]

$$= \frac{(\epsilon-1+log\alpha)\alpha e^{\epsilon-1}}{(\alpha e^{\epsilon-1}-1)}\sum_{i=0}^{m+n}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(\epsilon-1)^m(log\alpha)^n}{m!\,n!}\frac{\beta^i\theta^{i+2}}{(\theta\delta+\beta)^{i+1}}\binom{m+n}{i}\frac{\theta(1+m+n)\delta\Gamma(r+i+1)+\beta\Gamma(r+i+2)}{[\theta(1+m+n)]^{r+i+2}}$$

Therefore, r^{th} moment is

$$\mu_r^i = \frac{(\epsilon-1+log\alpha)\alpha e^{\epsilon-1}}{(\alpha e^{\epsilon-1}-1)}\sum_{i=0}^{m+n}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(\epsilon-1)^m(log\alpha)^n}{m!\,n!}\frac{\beta^i\theta^{i+2}}{(\theta\delta+\beta)^{i+1}}\binom{m+n}{i}\frac{\theta(1+m+n)\delta\Gamma(r+i+1)+\beta\Gamma(r+i+2)}{[\theta(1+m+n)]^{r+i+2}} \quad (18)$$

The mean of the newly transformed random variable is

$$E(X) = \mu_1^i$$

$$= \frac{(\epsilon-1+log\alpha)\alpha e^{\epsilon-1}}{(\alpha e^{\epsilon-1}-1)}\sum_{i=0}^{m+n}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(\epsilon-1)^m(log\alpha)^n}{m!\,n!}\frac{\beta^i\theta^{i+2}}{(\theta\delta+\beta)^{i+1}}\binom{m+n}{i}\frac{\theta(1+m+n)\delta\Gamma(i+2)+\beta\Gamma(i+3)}{[\theta(1+m+n)]^{i+3}} \quad (19)$$

The moment generating function of the newly suggested five parameters Lindley distribution is

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\, f(x;\theta,\delta,\beta,\alpha,\epsilon)dx$$

[from the power series expansion $e^{tx} = \sum_{p=0}^{\infty}\frac{(tx)^p}{p!}$

$]=\sum_{p=0}^{\infty}\frac{t^p}{p!}\int_0^\infty x^p f(x;\theta,\delta,\beta,\alpha,\epsilon)dx$

$=\sum_{p=0}^{\infty}\frac{t^p}{p!}\mu_p^i$

Where,

$$\mu_p^i = \frac{(\epsilon-1+log\alpha)\alpha e^{\epsilon-1}}{(\alpha e^{\epsilon-1}-1)}\sum_{i=0}^{m+n}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(\epsilon-1)^m(log\alpha)^n}{m!\,n!}\frac{\beta^i\theta^{i+2}}{(\theta\delta+\beta)^{i+1}}\binom{m+n}{i}\frac{\theta(1+m+n)\delta\Gamma(p+i+1)+\beta\Gamma(p+i+2)}{[\theta(1+m+n)]^{p+i+2}}$$

$$M_X(t) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\mu_p^i$$

$$= \sum_{p=0}^{\infty}\frac{t^p}{p!}\frac{(\epsilon-1+log\alpha)\alpha e^{\epsilon-1}}{(\alpha e^{\epsilon-1}-1)}\sum_{i=0}^{m+n}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(\epsilon-1)^m(log\alpha)^n}{m!\,n!}\frac{\beta^i\theta^{i+2}}{(\theta\delta+\beta)^{i+1}}\binom{m+n}{i}\frac{\theta(1+m+n)\delta\Gamma(p+i+1)+\beta\Gamma(p+i+2)}{[\theta(1+m+n)]^{p+i+2}} \quad (20)$$

4.2. The moments and moments-generating function of the newly suggested five parameters Lindley distribution can be obtained by simply putting $\boldsymbol{\alpha = 1}$ *,where* $\boldsymbol{\epsilon \neq 1}$.

The r^{th} moment of a random variable is used to determine the mean, variance, skewness and kurtosis and is generally defined as:

$$\mu_r^i = E(X^r) = \int_0^\infty X^r f(x;\theta,\delta,\beta,\epsilon)dx = \int_0^\infty X^r\frac{(\epsilon-1)\theta^2(\delta+\beta x)e^{-\theta x}e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}}{(e^{\epsilon-1}-1)(\theta\delta+\beta)}dx$$

$$= \frac{(\epsilon-1)\theta^2}{(\theta\delta+\beta)(e^{\epsilon-1}-1)}\int_0^\infty X^r e^{(\epsilon-1)[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}(\delta+\beta x)e^{-\theta x}dx$$

$$= \frac{(\epsilon-1)\theta^2 e^{\epsilon-1}}{(\theta\delta+\beta)(e^{\epsilon-1}-1)}\int_0^\infty X^r e^{(\epsilon-1)[-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}]}(\delta+\beta x)e^{-\theta x}dx \quad (21)$$

Wecanapply,

$$e^{-(\epsilon-1)\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}} = \sum_{k=0}^{\infty}\frac{(-1)^k(\epsilon-1)^k\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^k e^{-\theta k x}}{k!}$$

Now from (21) we get, μ_r^i=

$$\frac{(\epsilon-1)\theta^2 e^{\epsilon-1}}{(\theta\delta+\beta)(e^{\epsilon-1}-1)}\int_0^\infty X^r(\delta+\beta x)e^{-\theta x}\left[\sum_{k=0}^{\infty}\frac{(-1)^k(\epsilon-1)^k\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^k e^{-\theta k x}}{k!}\right]dx$$

$$=\frac{\theta^2 e^{\epsilon-1}}{(\theta\delta+\beta)(e^{\epsilon-1}-1)}\sum_{k=0}^{\infty}\frac{(\epsilon-1)^{k+1}(-1)^k}{k!}\int_0^\infty X^r(\delta+\beta x)\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)^k e^{-\theta(k+1)x}\,dx$$

$$=\frac{\theta^2 e^{\epsilon-1}}{(\theta\delta+\beta)(e^{\epsilon-1}-1)}\sum_{k=0}^{\infty}\frac{(\epsilon-1)^{k+1}(-1)^k}{k!}\int_0^\infty X^r(\delta+\beta x)\sum_{i=0}^{k}\binom{k}{i}\left(\frac{\theta\beta x}{\theta\delta+\beta}\right)^i e^{-\theta(k+1)x}\,dx$$

[By using series expansion]

$$=\frac{e^{\epsilon-1}}{(e^{\epsilon-1}-1)}\sum_{i=0}^{k}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\binom{k}{i}\frac{\beta^i\theta^{i+2}(\epsilon-1)^{k+1}}{(\theta\delta+\beta)^{i+1}}\int_0^\infty x^{r+i}(\delta+\beta x)e^{-\theta(k+1)x}\,dx$$

Therefore, r^{th} moment is, μ_r^i=

$$\frac{e^{\epsilon-1}}{(e^{\epsilon-1}-1)}\sum_{i=0}^{k}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\binom{k}{i}\frac{\beta^i\theta^{i+2}(\epsilon-1)^{k+1}}{(\theta\delta+\beta)^{i+1}}\frac{\theta\delta(k+1)\Gamma(r+i+1)+\beta\Gamma(r+i+2)}{[\theta(k+1)]^{r+i+2}} \quad (22)$$

The mean of the newly transformed random variable is

$$E(X)=\mu_1^i\frac{e^{\epsilon-1}}{(e^{\epsilon-1}-1)}\sum_{i=0}^{k}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\binom{k}{i}\frac{\beta^i\theta^{i+2}(\epsilon-1)^{k+1}}{(\theta\delta+\beta)^{i+1}}\frac{\theta\delta(k+1)\Gamma(i+2)+\beta\Gamma(i+3)}{[\theta(k+1)]^{i+3}} \quad (23)$$

The moment generating function of the newly suggested five parameters Lindley distribution is

$$M_X(t)=E(e^{tx})=\int_0^\infty e^{tx}f(x;\theta,\delta,\beta,\epsilon)dx$$

[from the power series expansion $e^{tx}=\sum_{q=0}^{\infty}\frac{(tx)^q}{q!}$]

$$=\sum_{q=0}^{\infty}\frac{t^q}{q!}\int_0^\infty x^q f(x;\theta,\delta,\beta,\epsilon)dx=\sum_{q=0}^{\infty}\frac{t^q}{q!}\mu_q^i$$

Where,

$$\mu_q^i=\frac{e^{\epsilon-1}}{(e^{\epsilon-1}-1)}\sum_{i=0}^{k}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\binom{k}{i}\frac{\beta^i\theta^{i+2}(\epsilon-1)^{k+1}}{(\theta\delta+\beta)^{i+1}}\frac{\theta\delta(k+1)\Gamma(q+i+1)+\beta\Gamma(q+i+2)}{[\theta(k+1)]^{q+i+2}}$$

$$M_X(t)=\sum_{q=0}^{\infty}\frac{t^q}{q!}\frac{e^{\epsilon-1}}{(e^{\epsilon-1}-1)}\sum_{i=0}^{k}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\binom{k}{i}\frac{\beta^i\theta^{i+2}(\epsilon-1)^{k+1}}{(\theta\delta+\beta)^{i+1}}\frac{\theta\delta(k+1)\Gamma(q+i+1)+\beta\Gamma(q+i+2)}{[\theta(k+1)]^{q+i+2}} \quad (24)$$

V. PARAMETER ESTIMATION

Let $X_1, X_2, X_3, \ldots\ldots\ldots, X_n$ be random observations with probability density function $f(x;\theta,\delta,\beta,\alpha,\epsilon)$. We construct likelihood function $L(\theta,\delta,\beta,\alpha,\epsilon;X)$ can be maximized using maximum likelihood estimation over the parameter domain, following [25,26]

5.1. The likelihood function of the newly suggested five parameters Lindley distribution with parameters $\boldsymbol{\epsilon,\alpha>0}$ & $\boldsymbol{\epsilon\neq\alpha\neq1}$ *can be expressed as follows:*

$$L(\theta,\delta,\beta,\alpha,\epsilon)=\prod_{i=1}^{n}f(x;\theta,\delta,\beta,\alpha,\epsilon)=\prod_{i=1}^{n}\frac{(\epsilon-1+\log\alpha)e^{(\epsilon-1)\left[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}\alpha^{\left[1-\left(1+\frac{\theta\beta x}{\theta\delta+\beta}\right)e^{-\theta x}\right]}\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x)e^{-\theta x}}{(\alpha e^{\epsilon-1}-1)} \quad (25)$$

The log likelihood function is expressed by

$$\ln L(\theta,\delta,\beta,\alpha,\epsilon)$$

$$=\ln(\epsilon-1+\ln\alpha)+(\epsilon-1)\sum_{i=1}^{n}\left[1-\left(1+\frac{\theta\beta x_i}{\theta\delta+\beta}\right)e^{-\theta x_i}\right]+\ln\alpha\sum_{i=1}^{n}\left[1-\left(1+\frac{\theta\beta x_i}{\theta\delta+\beta}\right)e^{-\theta x_i}\right]+\sum_{i=1}^{n}\ln\left[\frac{\theta^2}{\theta\delta+\beta}(\delta+\beta x_i)e^{-\theta x_i}\right]-\ln(\alpha e^{\epsilon-1}-1) \quad (26)$$

Then, by partial derivatives of equation (26) with respect to $\theta,\delta,\beta,\alpha,\epsilon$ and equating them to zeros, we obtain the following normal equations

$$\frac{\partial \ln L(\theta,\delta,\beta,\alpha,\epsilon)}{\partial \theta} = (\epsilon - 1)\sum_{i=1}^{n}\left(1+\frac{\theta\beta x_i}{\theta\delta+\beta} - \frac{\beta^2}{(\theta\delta+\beta)^2}\right)x_i e^{-\theta x_i} + \ln\alpha \sum_{i=1}^{n}\left(1+\frac{\theta\beta x_i}{\theta\delta+\beta} - \frac{\beta^2}{(\theta\delta+\beta)^2}\right)x_i e^{-\theta x_i} + \sum_{i=1}^{n}\left[\frac{(2-\theta x_i)}{\theta} - \frac{\delta}{\theta(\theta\delta+\beta)}\right] = 0 \quad (27)$$

$$\frac{\partial \ln L(\theta,\delta,\beta,\alpha,\epsilon)}{\partial \delta} = (\epsilon - 1)\sum_{i=1}^{n}\left(\frac{\theta^2\beta x_i e^{-\theta x_i}}{(\theta\delta+\beta)^2}\right) + \ln\alpha \sum_{i=1}^{n}\left(\frac{\theta^2\beta x_i e^{-\theta x_i}}{(\theta\delta+\beta)^2}\right)x_i e^{-\theta x_i} + \sum_{i=1}^{n}\frac{\beta(1-\theta x_i)}{(\theta\delta+\beta)(\delta+\beta x_i)} = 0 \quad (28)$$

$$\frac{\partial \ln L(\theta,\delta,\beta,\alpha,\epsilon)}{\partial \beta} = \frac{(\epsilon-1)\theta^2\delta}{(\theta\delta+\beta)^2}\sum_{i=1}^{n}\left(-e^{-\theta x_i}x_i\right) + \frac{(\ln\alpha)\theta^2\delta}{(\theta\delta+\beta)^2}\sum_{i=1}^{n}\left(-e^{-\theta x_i}x_i\right) + \frac{\delta}{\theta\delta+\beta}\sum_{i=1}^{n}\frac{(\theta x_i - 1)}{\delta+\beta x_i} = 0 \quad (29)$$

$$\frac{\partial \ln L(\theta,\delta,\beta,\alpha,\epsilon)}{\partial \alpha} = \frac{1}{\alpha(\epsilon-1+\ln\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n}\left[1-\left(1+\frac{\theta\beta x_i}{\theta\delta+\beta}\right)e^{-\theta x_i}\right] - \frac{e^{(\epsilon-1)}}{\alpha e^{(\epsilon-1)}-1} = 0 \quad (30)$$

$$\frac{\partial \ln L(\theta,\delta,\beta,\alpha,\epsilon)}{\partial \epsilon} = \frac{1}{(\epsilon-1+\ln\alpha)} - \frac{\alpha e^{(\epsilon-1)}}{\alpha e^{(\epsilon-1)}-1} = 0 \quad (31)$$

Solving equations (27)-(31) simultaneously, we get the estimators of $\theta, \delta, \beta, \alpha, \epsilon$. The Newton-Raphson method is used to get a solution to the above equations, as an analytical solution is not possible.

5.2. The likelihood function of the newly suggested four parameters Lindley distribution can be obtained by substituting $\boldsymbol{\alpha = 1}$ *, in those above equations, where* $\boldsymbol{\epsilon \neq 1}$*,*

Thus, we obtain the following normal equations

$$\frac{\partial \ln L(\theta,\delta,\beta,\epsilon)}{\partial \theta} = \sum_{i=1}^{n}\left[\frac{(2-\theta x_i)}{\theta} - \frac{\delta}{\theta(\theta\delta+\beta)}\right] + (\epsilon - 1)\sum_{i=1}^{n}\left(1+\frac{\theta\beta x_i}{\theta\delta+\beta} - \frac{\beta^2}{(\theta\delta+\beta)^2}\right)x_i e^{-\theta x_i} = 0 \quad (32)$$

$$\frac{\partial \ln L(\theta,\delta,\beta,\epsilon)}{\partial \delta} = \sum_{i=1}^{n}\frac{\beta(1-\theta x_i)}{(\theta\delta+\beta)(\delta+\beta x_i)} + (\epsilon - 1)\sum_{i=1}^{n}\left(\frac{\theta^2\beta x_i e^{-\theta x_i}}{(\theta\delta+\beta)^2}\right) = 0 \quad (33)$$

$$\frac{\partial \ln L(\theta,\delta,\beta,\epsilon)}{\partial \beta} = \sum_{i=1}^{n}\frac{\delta}{(\theta\delta+\beta)}\frac{(\theta x_i - 1)}{(\delta+\beta x_i)} + \frac{(\epsilon-1)\theta^2\delta}{(\theta\delta+\beta)^2}\sum_{i=1}^{n}\left(-e^{-\theta x_i}x_i\right) = 0 \quad (34)$$

$$\frac{\partial \ln L(\theta,\delta,\beta,\epsilon)}{\partial \epsilon} = \frac{1}{(\epsilon-1)} - \frac{(\epsilon-1)e^{(\epsilon-1)}}{e^{(\epsilon-1)}-1} = 0 \quad (35)$$

Solving equations (32)-(35) simultaneously, we get the estimators of $\theta, \delta, \beta, \epsilon$. The Newton-Raphson method is used to get a solution to the above equations, as an analytical solution is not possible.

VI. QUANTILE AND MEDIAN FUNCTIONS

6.1. The quantile and median functions of the newly suggested five parameters Lindley distribution with parameters $\boldsymbol{\theta, \delta, \beta, \epsilon, \alpha > 0}$ *&* $\boldsymbol{\epsilon \neq \alpha \neq 1}$ *can be expressed as follows:*

The quantile function of the newly suggested five parameters Lindley distribution of random variable X is $q_x(u) = F^{-1}(x)$

Consider, $p = G(X) = 1 - (1 + \frac{\theta\beta x}{\theta\delta+\beta})e^{-\theta x}$ (36)

$$u = F(x;\theta,\delta,\beta,\alpha,\epsilon) = \frac{e^{(\epsilon-1)p}\alpha^{p}-1}{(\alpha e^{\epsilon-1}-1)}$$

$$or, u(\alpha e^{\epsilon-1}-1)+1 = e^{(\epsilon-1)p}\alpha^{p}$$

Take logarithm for both sides we get, $p = \frac{\log[u(\alpha e^{\epsilon-1}-1)+1]}{\epsilon-1+log\alpha}$

(37)

Dividing both sides by β and simplifying the equation (38) we get

$$(p-1)\frac{\theta\delta+\beta}{\beta} = -(\frac{\delta\theta+\beta}{\beta} + \theta x)e^{-\theta x}$$

Multiplying both sides by $e^{-(\frac{\delta\theta+\beta}{\beta})}$ and simplifying we get

$$(p-1)\frac{(\theta\delta+\beta)e^{-(\frac{\delta\theta+\beta}{\beta})}}{\beta} = -\left(\frac{\delta\theta+\beta}{\beta} + \theta x\right)e^{-(\theta x+\frac{\delta\theta+\beta}{\beta})}$$

(38)

Using negative Lambert W_{-1} function from equation (40) we get

$$\frac{W_{-1}}{\beta}[(p-1)(\theta\delta+\beta)e^{-\left(\frac{\delta\theta+\beta}{\beta}\right)}] = -\left(\frac{\delta\theta+\beta}{\beta} + \theta x\right) or, x = -\frac{\delta}{\beta} - \frac{1}{\theta} - \frac{W_{-1}}{\theta}[\frac{(p-1)(\theta\delta+\beta)e^{-\left(\frac{\delta\theta+\beta}{\beta}\right)}}{\beta}]$$

(39)

The quantile function is, $q_x(u) = -\frac{\delta}{\beta} - \frac{1}{\theta} - \frac{W_{-1}}{\theta}[\frac{\left(\frac{\log(u(\alpha e^{\epsilon-1}-1)+1)}{\alpha} - \epsilon+1\right)(\theta\delta+\beta)e^{-\left(\frac{\delta\theta+\beta}{\beta}\right)}}{\beta(\epsilon-1+log\alpha)}]$

(40)

The median of the above-mentioned function should be as follows: [When $u = \frac{1}{2}$ in equation (41)]

$$median = -\frac{\delta}{\beta} - \frac{1}{\theta} - \frac{W_{-1}}{\theta}[\frac{\left(\frac{\log[(\frac{1}{2})(\alpha e^{\epsilon-1}-1)+1]}{\alpha} - \epsilon+1\right)(\theta\delta+\beta)e^{-\left(\frac{\delta\theta+\beta}{\beta}\right)}}{\beta(\epsilon-1+log\alpha)}$$

(41)

The 1st and 3rd quartiles are obtained by substituting $u = 0.25$ and $u = 0.75$, respectively.

6.2. The quantile and median functions of the newly suggested four parameters Lindley distribution can be obtained by putting $\boldsymbol{\alpha = 1}$ *in equation (40) where* $\boldsymbol{\epsilon \neq 1}$*,*

The quantile function is, $q_x(u) = -\frac{\delta}{\beta} - \frac{1}{\theta} - \frac{W_{-1}}{\theta}[\frac{(\log(u(\alpha e^{\epsilon-1}-1)+1) - \epsilon+1)(\theta\delta+\beta)e^{-\left(\frac{\delta\theta+\beta}{\beta}\right)}}{\beta(\epsilon-1)}]$

(42)

The median of the above-mentioned function should be as follows: [When $u = \frac{1}{2}$ in equation (41)]

$$median = -\frac{\delta}{\beta} - \frac{1}{\theta} - \frac{W_{-1}}{\theta}[\frac{\left(\log((\frac{1}{2})(\alpha e^{\epsilon-1}-1)+1) - \epsilon+1\right)(\theta\delta+\beta)e^{-\left(\frac{\delta\theta+\beta}{\beta}\right)}}{\beta(\epsilon-1)}]$$

(43)

The 1st and 3rd quartiles are obtained by substituting $u = 0.25$ and $u = 0.75$, respectively.**VII.**

VII. GRAPHICAL REPRESENTATION

Plots of *pdf* and *cdf* for different values of $\epsilon[\epsilon \in \{2,3,4,5,6\}]$, and constant values of other parameters $\theta = \delta = \beta = \alpha = 1.5$ are shown in Fig. 1. and Fig. 2. It is observed that the probability density function of this five parameters probability distribution is right-skewed, unimodal and a declining function. The *cdf* of all convergence to 1.0.Graph of *survival function* with different values of ϵ, and fixed values of other parameters asbeforeareshown in Fig. 3. It is observed that larger ϵ values give higher survival probability vice versa. The survival function for the new distribution exhibits a declining behavior over the domain *x* with different rates of decay. Fig. 4. shows the graph of hazard*function* for different values of $\epsilon \in \{2,3,4,5,6\}$, and constant values ofotherparameters $\theta = \delta = \beta = \alpha = 1.5$.This figure demonstrate that the new hazard function exhibits very interesting shapes and strictly increasing property.

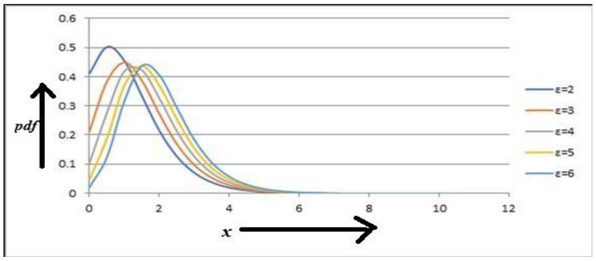

Fig.1. Plots of *pdf* with different values of $\epsilon \in \{2,3,4,5,6\}$ and same values of other parameters $\theta = \delta = \beta = \alpha = 1.5$

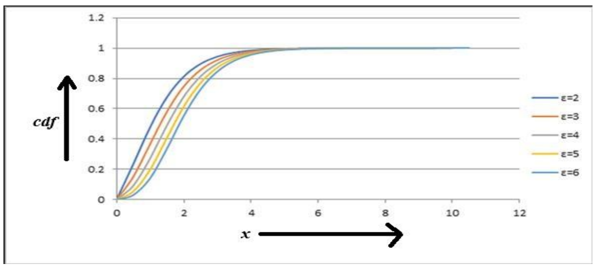

Fig. 2. Plots of *cdf* with different values of $\epsilon \in \{2,3,4,5,6\}$ and same values of other parameters $\theta = \delta = \beta = \alpha = 1.5$

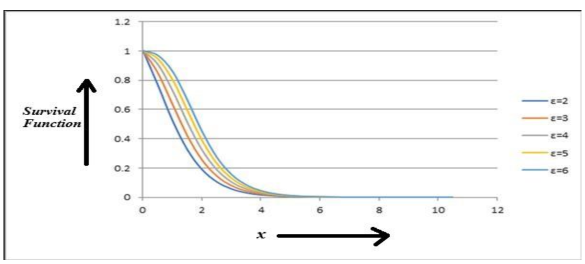

Fig. 3. Plots of *survival function* with different values of $\epsilon \in \{2,3,4,5,6\}$ and same values of other parameters $\theta = \delta = \beta = \alpha = 1.5$

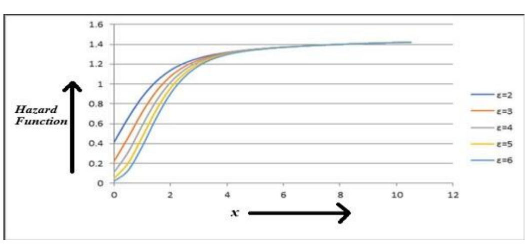

Fig. 4. Plots of hazard *function* with different values of $\epsilon \in \{2,3,4,5,6\}$, and same values of other parameters $\theta = \delta = \beta = \alpha = 1.5$

CONCLUSION

The Lindley probability distribution has been generalized in this paper up to five parameters. Thus, we introduced a new distribution in the basis of APT Lindley distribution. We developed new survival and hazard functions, cumulative distribution, probability density functions. Plots with various parameter values are another way in which they are demonstrated. Its properties including moments, moment-generating and quantile functions are examined in this paper. Our research work can also be used to solve a variety of problems in biomedical sector, industrial reliability, health and engineering sectors.

REFERENCES

[1] D.Lindley and V.Fiducial," Distributions and baye's theorem" Journal of the Royal Statistical Society: Series B, vol. 20, no. 1, pp. 102-107, (1958)

[2] Ghitany, M.E., Balakrishnan, N., Al-Enezi, L.J., Al-Mutairi, D.K.: Power Lindley distribution and associated inference. Comput. Stat. Data Anal. 64, 20-33 (2013)

[3] Shanker, R., Sharma, S., Shanker, R.: a two-parameter Lindley distribution for modelling waiting and survival times data (2013)

[4] H.S. Bakouch, BV.M. Al-Zahrani, A.A.Al-Shomrani, V.A.Marchi, and F.Louzada, "An extended Lindley distribution", Journal of the Korean Sergical society, Vol.41, no.1, pp. 75-85, (2012)

[5] S.Nadaraj, H.S.Bakouch, and R.Tahmasbi,"A generalized Lindley distribution,' Sankhya B, vol. 73, no. 2, pp. 331-359, (2011)

[6] D.Hamed and A.Alzaghal,"New class of Lindley distributions: properties and applications," Journal of statistical distributions and Applications, vol. 8, no.1, pp.11-22, (2021)

[7] A.Mahdavi and D.Kundu, "A new method for generating distribution with an application to exponential distribution," Communications in Statistics – Theory and methods, vol.46, no.13, pp. 6543-6547, (2017)

[8] M.E.Ghitany, B.Atieh and S.Nadarajah, "Lindley distribution and its application," Mathematics and Computers in Simulation,vol.78, no. 4, pp.493-506, (2008)

[9] A.Khalil, A.A.H.Ahmadini, M.Ali, W.K.Mashwani, S.S.Alshquq and Z.Salleh, “A novel method for developing efficient probability distributions with applications to engineering and life science data,” Journal of mathematics, vol. 2021, Article ID 4479270,13 pages , (2021)

[10] F.Y.Eissa and C.D.Sonar, “Alpha power transformed extended power Lindley distribution,” Journal of Statistical Theory and Applications, vol.22, no.1-2, pp. 1-18, (2022)

[11] Jones, M.C.: On families of distributions with shape parameters. Int. Stat. Rev. 83(2), 175–192 (2015)

[12] Dey, S., Nassar, M., Kumar, D.: “Alpha power transformed inverse Lindley distribution: a distribution with an upside-down bathtub-shaped hazard function,” J.Comput. Appl. Math. 348,130-145 (2019)

[13] S.J.Dugasa, A.T.Goshu and B.G.Arero: “Alpha Power Transformation of Lindley Probability Distribution” Wiley, A Journal of Probability and Statistics, Article ID 9068114, 18 pages, vol(2024)

[14] M.E.Ghitany, F.Alqallaf, D.K.Al-Mutairi and H.A.Husain, “A Two-parameter Lindley distribution and its applications to survival data,” Math Comput Simulat 81, 1190-1201, (2011)

[15] A.S.Hassan, R.E.Mohamad, M.Elgarhy and A.Fayomi, : “Alpha power transformation extended exponential distribution: properties and applications,” The Journal of Nonlinear Science and Applications vol. 12, no. 04, pp. 239-251, (2018)

[16] J.Mazucheli and J.A.Archer, “The Lindley distribution applied to competing risks lifetime data,” Comput Meth Prog Bio 104, 188-92, (2011)

[17] M. Ahsan-ul-Haq, S.M Choudhary, A.H.Al-Marshadi and M.Aslam, “A new generalization of Lindley distribution for modelling of wind speed data,” Energy Reports, vol.8, no.1, pp. 1-11, (2022)

[18] R.A.ZeinEldin, M.A.U.Haq, S.Hashmi and M.Elsehety, “Alpha power-transformed inverse Lomax with different methods of estimation and applications,” Complexity, vol. 27, pp. 1-5, (2020)

[19] Hassan, A.S., Elgarhy, M., Mohamd, R.E., Alrajhi, S.: “On the alpha power transformed Lindley distribution,” J. Probab. Stat. (2019)

[20] Eltehiwy, M.: “On the alpha power transformed power inverse Lindley distributon,” J. Indian Soc, Probab. Stat. 1-24 (2024)

[21] S. Dey, V. K. Sharma, and M. Mesfioui, “A new extension of Weibull distribution with application to lifetime data,” *Annals of Data Science*, vol. 4, no. 1, pp. 31–61, (2017).

[22] S. Dey, A. Alzaatreh, C. Zhang, and D. Kumar, “A new ex- tension of generalized exponential distribution with appli- cation to ozone data,” Ozone: Science and Engineering, vol. 39, no. 4, pp. 273–285, (2017).

[23] S. Dey, I. Ghosh, and D. Kumar, “Alpha power transformed Lindley distribution: properties and associated inference with application to earthquake data,” Annals of Data Science, vol. 6, no. 4, pp. 623–650, (2019).

[24] Marshall, A.W., Olkin, I.: “A new method for adding a parameter to a family of distributions with application to the exponential and weibull families,” Biometrika 84(3), 641–652 (1997)

[25] A. F. Karr, Probability, Springer Science and Business Media, New York, NY, USA, (1993).

[26] G. G. Roussas, “Introduction to Probability and Statistical Inference,” Elsevier Science, Amsterdam, Netherlands, (2003).

[27] C. P. Robert and G. Casella, Introducing Monte Carlo Methods with R, Springer, Berlin, Germany, (2010).

[28] M. Aldeni, C. Lee, and F. Famoye, “Families of distributions arising from the quantile of generalized lambda distribution,” Journal of Statistical Distributions and Applications, vol. 4, no. 1, p. 25, (2017).

[29] Cordeiro, G.M., de Castro, M.: “A new family of generalized distributions,” J. Stat. Comput. Simul.81(7), 883–898 (2011)

[30] Alzaatreh, A., Lee, C., Famoye, F.: “A new method for generating families of continuous distributions,” Metron 71(1), 63–79 (2013)

[31] Lee, C., Famoye, F., Alzaatreh, A.Y.: "Methods for generating families of univariate continuous distribution in the recent decades," Wiley Interdiscip. Rev. Comput. Stati. 5(3), 219–238 (2013)