Exploring Special Curvatures Connections in Finsler Geometry

LAVKUSH PANDEY[1], S. RUHELA[2]
[1, 2]Department of Mathematics, M.L.K. P.G. College Balrampur, U.P.

Abstract- This study examines the advantages of a particular connection in Finsler geometry that addresses the equivalence problem. The connection considered is torsion-free and exhibits an "almost" compatibility with the associated metric. We focus on an intrinsic investigation of a specific category of regular Finsler connections within the framework of special curvatures and connections. As a result, this leads naturally to a generalized notion of sectional curvature in Finsler geometry, along with several related comparison theorems.

I. INTRODUCTION

Riemannian geometry has long served as the foundation for describing curved space-time. Its role in formulating and interpreting gravity is closely tied to the assumption of local isotropy in space-time. The metric structure of this geometry successfully supports the observational predictions of general relativity, such as the existence of geodesics and their deviations. Within the scope of physical geometry—concerned with analyzing space in relation to moving and interacting objects—the geometrical structure of space-time becomes central. In this setting, gravitational phenomena can be represented through geometrical entities like curvature, which provides a powerful tool for investigating the nature of the universe.

A natural extension arises when one incorporates not only positional information but also the direction or velocity of a particle into the framework of geometry. This leads to a generalization of the Riemannian metric known as Finsler geometry, sometimes referred to as the geometry of the variational calculus. Unlike Riemannian geometry, Finsler geometry is capable of describing locally anisotropic space-times, where many physical processes cannot be captured by the Riemannian model alone. Over the past decade, there has been growing scientific interest in the applications of Finsler, Finsler-like, and Lagrange-Finsler geometries. This renewed focus has significantly advanced their use in gravitation, general relativity, and cosmology, with several influential works contributing to this development.

Applications in Various Topics of Science

Beyond gravitation, Finsler geometry has found applications across diverse branches of science. In mathematical biology, for example, Peter Antonelli employed it to study problems in ecology, social interactions, predator–prey dynamics, and evolutionary theory. In the field of information thermodynamics, researchers such as Roman Ingarden and R. Mrugala introduced the notion of relative information (entropy) to establish a Riemannian structure on the space of thermodynamic parameters; for nonequilibrium systems, however, this structure naturally extends to a Finslerian one. Further applications appear in various areas of optics—including crystal, physiological, and electron optics. In seismology, Finsler geometry has been used to model seismic wave fronts in anisotropic media, where the wave front deviates from a circular form and instead assumes a convex curve known as a superellipse.Applications in Gravitation

The study of geodesic deviations within the framework of the tangent bundle has played a central role in the development of Raychaudhuri equations for congruences on the Finsler tangent bundle, particularly in the context of F–R spacetimes. Alternative approaches have also been proposed, and notable progress has been achieved in formulating classical physical fields on Finsler spacetimes. Several attempts to construct field equations for the generalized metric tensor are documented in the literature.

In this setting, the tangent Lorentz bundle emerges as a differentiable eight-dimensional manifold that can be endowed with a generalized metric-compatible connection and its associated curvature tensor. Consequently, it has become common practice to formulate an eight-dimensional analogue of the Einstein field equations on this bundle, employing these generalized quantities in place of the classical ones. An alternative perspective is provided by the osculating Riemannian space approach, which effectively reduces a Finslerian spacetime to a corresponding "Riemannian" model, thereby offering another route to address this problem.

Special Connections and Curvature

1. Connections:

A connection on a Finsler manifold (M,F) is defined as a linear connection on the vector bundle ρ: $\pi^*TM \to TM_0$ and it can be introduced through several approaches. Distinguished geometers such as É. Cartan and S. S. Chern have made significant contributions to this field. Of particular importance is the connection for Finsler metrics first introduced by S. S. Chern in 1943, which is now known as the Chern connection. Independently, H. Rund later presented the same connection in a different framework. Consequently, in some parts of the literature ([101], [11], p.171), the Chern connection is also referred to as the Rund connection.

1.1 Chern Connection

Let (M, F) be an n-dimensional Finsler manifold, and denote by TM0 = TM \{0} its slit tangent bundle. The natural projection π: $TM_0 \to M$ induces a vector bundle π^*TM on TM_0, whose fiber at a point (x, y) ∈ TM_0 is

$$\pi^*TM|(x,y) := \{(x, y, v)|v \in TxM\} \sim= T_xM$$

In other words, π^*TM is a vector bundle of rank nnn with base space TM_0. Its dual bundle is denoted by π^*T^*M, where the fiber at $(x,y)\in TM_0$ is the dual space T^*_xM corresponding to the fiber T_xM.

Theorem 1 (S. S. Chern). Let (M,F) be an nnn-dimensional Finsler manifold. On the pullback tangent bundle π∗TM, there exists a unique linear connection ∇ that is torsion-free and almost metric-compatible. More precisely, for any local frame field {ei} on π∗TM with corresponding dual coframe $\{\omega^i\}$ on π^*T^*M, there exists a unique collection of local 1-forms $\{\omega^i_j\}$ on TM_0 such that

$$d\omega^i - \omega^j \wedge \omega^i_j = 0, \tag{1.1}$$

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2C_{ijk}\omega^{n+k}, \quad \omega^{n+k} := dy^k + y^j\omega^k_j, \tag{1.2}$$

where $g_{ij}\omega^i \otimes \omega^j$ is the fundamental form, $C_{ijk}\omega^i \otimes \omega^j \otimes \omega^k$ is the Cartan tensor and $y = y^i e_i \in TM_0$ is a tangent vector.

(1.1) and (1.2) can be also written as

$$\nabla_u v - \nabla_v u = [u, v], \quad \forall u, v, w \in TM_0,$$
$$w(\langle u, v\rangle_y) - \langle \nabla_w u, v\rangle_y - \langle u, \nabla_w v\rangle_y = 2\mathbf{C}_y(\nabla_w y, u, v).$$

Remark. Since the Chern connection is constructed on the pullback tangent bundle π∗TM, it should be emphasized that $\nabla_u v$ and [u,v] in the preceding formulas must be interpreted as $\nabla_u(\rho^{-1}v)$ and $[\rho^{-1}u,\rho^{-1}v]=\rho^{-1}[u,v]$, according to the commutative diagram. The remaining expressions follow in the same way and are therefore not explicitly stated.

Proof of Theorem 1.1. Without loss of generality, the theorem may be proved in a standard local coordinate system (x^i,y^i) on TM_0. In this setting, the local frame field of the pullback tangent bundle π∗TM is given by $\{\partial_i=\partial/\partial x_i$, and its corresponding dual frame field in the dual bundle π∗T∗M is $\{dx^i\}$.

2. Curvature:

2.1 Curvature form of the Chern connection

Let (M, F) be an nnn-dimensional Finsler manifold. For any local frame field $\{e_i\}$ on the pullback tangent bundle π^*TM and its dual coframe $\{\omega^i\}$ on the dual bundle π^*T^*M, the Chern connection forms $\{\omega^i_j\}$ are determined by (1.1) and (1.2). The corresponding curvature 2-forms are then defined by

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k. \tag{2.1}$$

Equivalently,

$$\left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)Z := \Omega(X,Y)Z$$

for $X, Y, Z \in \mathcal{C}(\pi^*TM)$.

Exterior differentiating (1.1) and using (1.1) and (2.1), one can obtain

$$\begin{aligned} 0 &= d^2\omega^i = d\omega^j \wedge \omega^i_j - \omega^j \wedge d\omega^i_j \\ &= -\omega^j \wedge \Omega^i_j \end{aligned}$$

(2.2)

which is called the first Bianchi identity.

CONCLUSION

In the coming years, further developments in the applications of Finsler and Finsler-like geometries are anticipated, particularly in the areas of general relativity, gravitation, and cosmology, where they may provide valuable tools for addressing problems related to a universe with weak anisotropic fields in these domains.

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