Existence and Uniqueness of Optimal Control for Fractional Nonlinear Systems. Please Send the Topic to Them

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Abstract- Necessary and sufficient conditions for the existence, form and uniqueness of optimal control of nonlinear systems are obtained. By using a generalized form of the open mapping theorem, controllability results are obtained for the free system by subjecting certain smoothness conditions on the nonlinear base. It is further shown that, if the system is relatively controllable, then optimal control is unique and bang-bang.

I. INTRODUCTION

Controllability is one of the fundamental concepts in mathematical systems theory. It is a qualitative property of dynamical control systems and it is of particular importance to the control theorist. As interpreted by the theory of nonlinear ordinary differential equations, [ ], the fundamental problem of control theory has been mathematically posed and answered. The authors, as a result, believe that a thorough and careful presentation of the current status on trends in the optimal control of nonlinear neutral systems will serve the useful purpose of offering a foundation on which later researches could be based

We are not only interested in the controllability of phenomena, but also in reaching the desired target with minimum wastage of efforts and in optimal time. It is this pursuit that has given rise to optimal control problems.

In this work, we shall consider the system:

$$\frac{d}{dt}D(t,x_t) = Ax(t-h) + F\big(x(t)\big) + \sum_{i=1}^{N} B_i u(t-h_i)] \qquad (1.1)$$

Optimal controls are used in determining optimal ways in controlling dynamical systems. Most theoretical works in this field serve as a foundation for our research. We here consider some recent works in the literature that are relevant to our study: Eyisi et al (2019) worked on Mathematical models for optimization of grid-integrated energy storage systems. The paper reviews mathematical models for optimization of grid-integrated energy storage systems. They discussed about optimization models including linear programming, mixed-integer linear programming and dynamic programming. They also presented case studies and simulation results to demonstrate the effectiveness of the proposed model.

On their work in the existence results for fractional neutral functional differential equations with infinite delay and nonlocal boundary conditions, Madeaha A. and Shahad A. established sufficient criteria for ensuring the existence of solutions and uniqueness for a class of nonlinear neutral Caputo fractional differential equations supplemented with infinite delays and nonlocal boundary conditions involving fractional derivatives. The theory of infinite delay and standard fixed-point theorems are employed to obtain the existence results for the given problem. Examples were constructed to illustrate the obtained results.

Nadjet (2023) on his work on Existence and controllability results for nonlocal fractional impulsive differential inclusions in Banach Spaces dealt with the existence of mild solutions for nonlocal fractional impulsive semilinear differential inclusions involving Caputo derivatives in Banach Spaces in the case when the linear part is the infinitesimal generator of a semigroup not necessarily compact. Meanwhile, they prove the compactness property of the set of solutions. Secondly they established two cases of sufficient conditions for the controllability of the considered control problems.

Zhang (2020) on his paper, Optimization methods based on optimal control transformed the optimization problem into the optimal control problem by designing an appropriate cost function. The iterative update gain for the optimization is derived using Pontryagin's Maximum Principle and the associated forward-backward difference equations (FBCE's).

K.S. Nisar et al (2024) treated the optimal and total controllability approach of non-instataneous Hilfer fractional derivative with integral boundary conditions. The focus was on absolute controllability of Hilfer impulsive non-instantaneous neutral derivative (HINND) with integral boundary condition of any order. Total controllability refers to the system's ability to be controlled during the complete interval of the process. Optimal control was thus achieved by shooting at a particular tgarget within a given minimum time in the interval.

On their work on Optimal control of linear hereditary systems with quadratic criterion, Clement and Walters (1992) addressed systems governed by retarded functional differential differential equations and provided an exact solution for a differential-delay equation with one delay. They formulated the optimal control problem with quadratic cost and expressed the solution in terms of an M2 operator, H(t) which is solved using the Rocatti differential operator equation. This paper also explores approximation techniques based on eigen functions for some complex scenarios.

Stability and optimal control theory of hereditary systems with applications was studied by Ethelbert (1992). This work derived equations for the dynamics of deterministic models including oscillating flying vehicles, mechanical systems and robotics. It investigated optimal feedback control strategies and integrated robotics into mathematical model to describe mathematical model to describe the optimal of dynamic systems.

Approximating the linear quadratic optimal control law for hereditary systems with delays in the control was done by Mark (1988). In his work, he described the approximation schemes for freedback controllers in distributed parameter systems with control delays. It introduces a factorization approach for deriving approximations to optimal feedback gains and presents two algorithms, including a fast algorithm for time-invariant cases supported by numerical examples schemes tor feedback controllers. It also introduces a factorization approach for deriving approximations to optimal control feedbacks.

II. NOTATIONS AND PRELIMINARIES

In this paper, the state space will be a continuous function, $C([-h,0],E^n)$ from $[-h,0]\ \ to\ \ E^n$ with the topology of uniform convergence. The norm of $\varphi \epsilon C([-h,0],E^n)$ is given by

$$\|\varphi\| = \operatorname*{Sup}_{-h\leq\varphi\leq 0} |\varphi(\theta)| \qquad (2.1)$$

Constrained control set will be a closed and bounded subset of U with values in C^m given by

$$C^m = \{u: u \in E^n,\ u_j \leq 1\}$$

With h>0, if $x: [-h, t_1] \to E^n$ and $t \to [0, t_1]$, the symbol x_t denotes a function on $[-h,0]$ with $x_t(s) = x(t+s), \quad s \in [-h,0]$

Consider the system of interest given by equation (1.1):

$$\frac{d}{dt}D(t,x_t) = Ax(t-h) + F\big(x(t)\big) + \sum_{i=1}^{N} B_i u(t-h_i)]$$

For $t \in [0,T], T > h$ with zero initial conditions $x(t) = \varphi(t), x(0) = 0, u(t) = 0$ for $t \in [-h,0]$

Here D is a bounded linear operator from $C\{[-h,0],E)$ into E^n defined by

$$D(t,x_t) = x(t) - A_{-1}x(t-h) \qquad (2.2)$$

The state $x(t) \in E^n = X$ and the control $u(t) \in E^m = U$

A is an $n \times n$ dimensional constant matrix. $B_i, i = 0,1,2,\dots,N$ are $n \times m$ dimensional constant matrices. $0 = h_0 < h_1 < h_2 < \cdots < h_N = h$ are constant delays. We let C = $C\{[-h,0]E^n)$be the Banach space of continuously differentiable functions. Let $E(-\infty,\infty)$ be the real line. For any integer n, E^n is the Euclidean space of n – tupples with the norm ‖·‖.

Let $L_1([0,T],E^n)$ be the space of Lebesque integrable functions taking [0,T] into E^n with norm $\|\varphi\| = \int_0^T |\varphi(x)|ds,\ \varphi \in L_1$. $L_\infty([0,T],E^m)$ is the space of essentially bounded functions taking [0,T] into E^m with norm

$\|\varphi\| = \operatorname*{Sup}_{0\leq s\leq T} |\varphi(s)|$for $\varphi \in L_\infty$.

The solution of system (1.1) is given by

$$x(t,\varphi,u) = S(t,\varphi,0)\varphi(t_0) + \int_0^t S(t-s)\Big(F\big(x(s,u)\big)\Big) + \sum_{i=1}^{N} B_i u(t-h_i)]ds \quad (2.3)$$

Here S(t) is an nxn transition matrix for the free system of (1.1) defined by

$$\frac{d}{dt}D(t,x_t) = x(t) - A_{-1}x(t-h) \qquad (2.4)$$

We introduce certain notations and present some important facts from the general theory of controllability that will be useful in this work.

Let U and X be given spaces and $g(u): U \to X$ be a mapping continuously differentiable near the origin of U. let us suppose for convenience that $g(0) = 0$. It is well known from the implicit function theorem that if the derivative

$Dg(0): U \to X$

maps the space of U onto the whole space X, then the nonlinear map transforms a neighbourhood of zero from the space of U onto some neighbourhood of zero into the space X.

Now, let us consider the more general case when the domain of the nonlinear operator g is in Ω, an open subset of U containing zero. Let U_c denote a closed and convex cone in U with vertex at zero. In the sequel, we shall use for controllability investigations, some properties of the nonlinear mapping, g which is a consequence of a generalized open mapping theorem. This result seems to be widely known. For the sake of completeness, we shall present it here without proof and in a slightly less generalized form sufficient for our purpose.

We now prove the convexity and compactness of the set functions, the Reachable and Attainable sets. However, we shall first establish a relationship between the two set functions to enable us see that, once the properties have been proved for one set, then they are applicable to the other set, then they are applicable to the other set. From equation (1.5), taking

$$x(t,u) = \check{A}(t,u)(t,u)$$

we have

$\check{A}(t,u) = \mathcal{S}(t,\varphi,0)\emptyset(t) + \int_0^t \mathcal{S}(t - s)\left(F\left(x(s,u)\right)\right) + \sum_{i=0}^{N} B_i u(t-h_i)]$

$= \mathcal{S}(t,\varphi,0)[\emptyset(t_0) + \Re(t)]$

Thus $\check{A}(t,u) = \mathcal{S}(t,\varphi,0)[\emptyset(t_0) + \Re(t)]$

We shall use the attainable set to establish that the two set functions possess the properties of convexity, closedness and compactness.

Definition 2.1: (Complete State)

The complete state of a differential system is defined as

$$y_t = \{x(t), x_t, u_t\} \quad (2.5)$$

where $u_t(\theta) = u(t_t\theta), \theta \in [-h,0]; h > 0$

Definition 2.2: (Controllability)

A system is said to be controllable on $[t_0, t_1]$ if every complete state, $y(t)$, and every $x_1 \in E^n$, there exists a control $u \in U$ such that the corresponding trajectory of the system satisfies $x(t_1) = x_1$

Definition 2.3 : (Optimal control)

u^* is an optimal control if there exists $t^* = Inf\{t: \tilde{A}(t,x) \cap G(t,u) \neq \varphi\}$ and

$u^* = Inf\{u: \tilde{A}(t,x) \cap G(t,u) \neq \varphi\}$

III. MAIN RESULTS

In this section, we shall proceed in showing that the system under study is controllable. Thereafter, we shall be interested in determining the minimum control energy for the pursuit of our target in order to capture it. The first statement raises and answers the question of controllability while the second addresses the optimal control problem, which is the focus of this chapter.

The following theorem provides a major result for the sufficient condition of the system to be controllable.

Theorem 3.1: (Relative controllability)

Suppose

i) $F(0) = 0$
ii) $U_c \in U$ is a closed and convex cone with vertex at zero.
iii) The associated linear control system with multiple delays in the control is U_c relatively controllable in [0,T].

Then the nonlinear neutral system (1.1) with multiple delays in the control is U_c relatively controllable in [0,T].

Proof: Let us define for the nonlinear neutral system (1.1) a nonlinear map

$$g: L_\infty([0,T], U_c) \to X$$

given by $g(u) = x(T,u)$

Similarly, for the associated linear dynamical neutral system (1.1), we define a linear map, $H: L_\infty([0,T], U_c) \to X$ by $Hv = x(T,v)$.

By the assumption (iii), the associated linear neutral system is U_c −relatively controllable in [0,T]. Therefore, the linear operator, H, is surjective, that is, it maps cones of admissible controls U_{ad} into the whole space. Furthermore, by lemma we have that

Dg(0)=H. Since U_c is a closed and convex cone, then the cone is of admissible controls.

$U_{ad} = L_\infty([0,T], U_c)$ is also a closed and convex cone in the function space, $L_\infty([0,T], U_c)$. Therefore the nonlinear map, g, transforms a conical neighborhood of zero in the cone of admissible controls, U_{ad} into some neighborhood of zero in the whole space, X. This is equivalent to the relative controllability in [0,T] of the nonlinear neutral control system (1.1) Hence, our Theorem follows.

In practical applications, of Theorem 4.1, the most difficult problem is to verify the assumption (iii) of the constrained controllability of neutral systems. In order to avoid this serious disadvantage, we use the following corollary:

Corollary 3.1:

Suppose the set U_c is a cone with vertex at zero and a nonempty interior in E^n, then the associated linear neutral system of (1.1) is U_c −relatively controllable in [0.T] if and only if :

(i) It is relatively controllable without any constraints. That is

$$Rank[B_0, B_1, \dots, B_N, HB_0, HB_1, \dots, HB_N, H^2B_0, H^2B_1, \dots, H^2B_N, \dots, H^{N-1}B_N] = n$$

(ii) there is no real eigenvector $v \in E^n$ of the matrix H satisfying
$vB_N \leq 0$ for all $U_c \in U$

Let us observe that, for a special case where $T < h_1$, relative controllability problem in $[0,T]$ for neutral systems with delays in the control may be reduced to the well-known standard controllability problem for dynamical control systems without delays in the control.

Corollary 3.2:
Suppose $T < h_1$ and the assumptions of Theorem 3.1 are satisfied, then the associated linear neutral control system of (1.1) is U_c −relatively controllable in [0,T] if and only if is controllable without constraints, that is

$$Rank[B_0, HB_0, H^2B_0, H^{N-1}B_0] = n$$

(i) and there is no real eigenvector eigenvector $v \in E^n$ of the matrix H satisfying
$vB_N \leq 0$ for all $U_c \in U$

Constrained controllability for nonlinear neutral systems is achieved by making use of the generalized open mapping theorem. It is shown that, if U is a closed and convex cone with vertex at zero, then the nonlinear neutral control system with multiple point delays in the control is U_c −relatively controllable in [0,T].

Theorem 3.2:
Let u^*be the optimal control for the neutral control system (1.1) with minimum time, t^*, then the target $g(t) = x(t^*, x_0, u^*)$ is on the boundary of the attainable set, $\check{A}$. That is, $g(t) \in \partial\check{A}$, where ∂ denotes the boundary.

Proof:
Suppose u^* is the minimum control -energy, then, by the relative controllability of system (1.1)
$g(t) = x(t^*, x_0, u^*) = x(t^*, t_0)[\eta + y^*]$; $y^* \in \Re(t^*, t_0)$ (see expression 3.16)

Therefore,$x(t^*, x_0, u^*) \in \Re(t^*, t_0)$ and consequently, $x(t^*, x_0, u^*) \in \check{A}(t^*, t_0)$

Suppose $x(t^*, x_0, u^*)$ is not on the boundary of $\check{A}(t^*, t_0)$, then it is in its interior for $t^* > t_0$.

Therefore there exists a ball $B(g(t^*), r)$ of radius r about the target, $g(t^*)$, such that : $B(g(t^*), r) \in \check{A}(t^*, t_0)$

Since $\check{A}(t^*, t_0)$ is a continuous set function of t, we can preserve the above inclusion for t near t^* if we reduce the size of the ball $B(g(t^*), r)$. That is, if there exists $\varepsilon > 0$ such that $B\left(g(t^*), \frac{1}{2}\right) \subset \check{A}(t^*, t_0)$; $t^* - \varepsilon \leq t \leq t_1$

Thus, $g(t^*) \in \check{A}(t^*, t_0)$ *for all* $t^* - \varepsilon \leq t \leq t_1$

This contradicts the optimaility of t^* and t^* as the minimum energy respectively. This contradiction however proves that g(t) is on the boundary of the attainable set, that is, $g(t) \in \partial\check{A}(t^*, t_0)$

Theorem 3.3 (Existence of optimal control):
Consider system (1.1) with its basic assumptions. Suppose the system is relatively controllable at time $t_1 > t_0$, then there exists an optimal control.
Proof:.
Let $g(t) \in G(t_1, t_0)$. By the assumption of relative controllability for the system,

$$g(t) \in G(t_1, t_0) \cap \check{A}(t_1, t_0) \neq \emptyset$$

So $g(t) \in \check{A}(t_1, t_0)$. Since $\check{A}(t_1, t_0)$ is a translation of the reachable set through $\eta \in E^n$,

$$\eta = \varphi(t_0) \sum_{i=0}^{N} \int_{t_0-h}^{t_0} S(t_{0,}, s)[A(s)x(s) + B_i(s+h_i)u]ds \quad (3.1)$$

We have, $g(t) \in \Re(t_1, t_0)$ and so

$$g(t) = \int_{t_0}^{t_1} z(t_0, s)u(s)ds \quad (3.2)$$

Let $t^* = Inf\{t: g(t) \in \Re(t_1, t_0)\}$

Now, $t_0 \le t \le t_1$ and there is a non-increasing sequence of time t_n. That is

$$t_n \le t_{n-1} \le t_{n-2} \le t_{n-3} \le \cdots \le t_1 \le t_0$$

and a sequence of controls $u_n \in U$.

Let $g(t) = y(t,u) \in \Re(t_1. t_0)$. That is $g(t_0) = y(t_0, u_0) \in \Re(t_1. t_0)$.

Also, $|g(t^*) - y(t^*, u_0)| \le |g(t^*) - g(t_0)| + |g(t_0) - y(t^*, u_0)|$

$$\le |g(t^*) - g(t_0)| + |y(t_0, u_0) - y(t^*, u_0)|$$

$$\le \left| g(t^*) - g(t_0) + \int_{t^*}^{t_n} \|y(s, u_n)\| \, ds \right|$$

By the continuity of $g(t)$ and the integrability of $\|y(t, u(t))\|$, it follows that, $y(t^*, u_0)$ tends to $g(t^*)$ as t tends to infinity. Since $R(t_1. t_0)$ contains $y(t^*, u_0)$ for each y and $\Re(t_1. t_0)$ is closed, then, $g(t^*) \in y(t^*, u^*)$ for some $u^* \in U$ and by definition 1.6, $t^* and\ u^*$ are optimal and $u^*$is the optimal control. This establishes the existence of the optimal control of the nonlinear neutral system (1.1)

Theorem 3.4 (Form of the optimal control):
Consider system (3.1) with its basic assumptions, u^* is the optimal control if and only if it is of the form

$$u^* = Sgn\, [c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)] \quad (3.3)$$

where $c \in E^n$.

Proof:
Suppose u^* is the optimal control of the system (3.1), then it maximizes the rate of change of

$y(t, u) = c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)\, u(s)$

In the direction of the vector c, that is , we want to maximize

$$c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)\, u(s)$$

Since $u(t)$ is an admissible control, that is, they are constrained to lie in a unit sphere, we have

$$|c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)\, u(s)| \le |c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i) * 1| \quad (3.4)$$

Applying the definition of a signum function, equation (4.4) is given below

$$c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i) \le c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i) * Sgn\, c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i) \quad (3.5)$$

Defining the signum function by u^*, we have (4.5) given as

$$c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i) \le c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)\, u^* \quad \text{for all } t > t_0$$

This means that, the optimal control energy, u^*, has the form

$u^* = Sgn\, [c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)]$ as in equation (3.3)

Conversely let

$$u^* = Sgn\, [c^T S(t_0, s) \sum_{i=0}^{N} B_i(s + h_i)]$$

Integrating (4.4) for all admissible control $u_{ad} \in U$, we have

$$\left| \int_{t_0}^{t_1} c^T S(t_0, s) \sum_{i=0}^{N} B_i(s + h_i)u(s)\, ds \right| \le \int_{t_0}^{t_1} |\sum_{i=0}^{N} c^T S(t_0, s)B_i(s + h_i)\, u^*(s)| ds$$

$$\le \int_{t_0}^{t_1} |\sum_{i=0}^{N} c^T S(t_0, s)B_i B_i(s + h_i) Sgn\, [c^T \sum_{i=0}^{N} S(t_0, s)(s + h_i)]| ds$$

This shows that u^* maximizes $c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)$ over all admissible controls. Hence, it is the optimal control.

Theorem 3.5 (Uniqueness of the optimal control)

Consider the system (1.1) with its basic assumptions. If u^* is the minimum control

Energy that brings the system to its target, then $u^*$is unique.

Proof:

Let u^* and v^* be two optimal controls that bring system (1.1) to its target, then both u^* and v^* maximizes $c^T \sum_{i=0}^{N} S(t_0, s)B_i(s + h_i)$ for some $c \in E^n, c \ne 0$ over all admissible controls and so,

$$\left| \int_{t_0}^{t_1} c^T S(t_0, s) \sum_{i=0}^{N} B_i(s + h_i)u(s)\, ds \right| \le \int_{t_0}^{t_1} |\sum_{i=0}^{N} c^T S(t_0, s)B_i(s + h_i)\, u^*(s)| ds \quad (3.7)$$

Also, using v^*as the optimal control, we have

$$\left|\int_{t_0}^{t_1} c^T S(t_0,s) \sum_{i=0}^{N} B_i(s+h_i)u(s)\,ds\right| \leq \int_{t_0}^{t_1} \left|\sum_{i=0}^{N} c^T S(t_0,s) B_i(s+h_i)\, v^*(s)\right| ds \quad (3.8)$$

Rewriting (4.7) and (4.8) in the form of equation, we have

$$\max_{-1\leq u\leq 1} \int_{t_0}^{t_1} |c^T S(t_0,s) B_i(s+h_i) u(s)| ds = \int_{t_0}^{t_1} |c^T S(t_0,s) B_i(s+h_i) u^*(s)| ds \quad (3.9)$$

$$\max_{-1\leq u\leq 1} \int_{t_0}^{t_1} |c^T S(t_0,s) B_i(s+h_i) u(s)| ds = \int_{t_0}^{t_1} |c^T S(t_0,s) B_i(s+h_i) v^*(s)| ds \quad (3.10)$$

Combining equations (3.9) from (3.10) gives

$$\int_{t_0}^{t_1} |c^T S(t_0,s) B_i(s+h_i)[u^*(s) - v^*(s)]| ds \quad] = 0$$

This can only be possible if $u^*(s) - v^*(s) = 0$ whuich implies $u^*(s) = v^*(s)$ and proving uniqueness of the optimal control.

CONCLUSION

Relative controllability for nonlinear neutral systems can be achieved by making use of the generalized open mapping theorem. It is shown that if U is a closed and convex cone with vertex at zero, then the nonlinear neutral dynamical control system with multiple delays in the control is relatively controllable in [0,T]. In the foregoing, we formulated and proved sufficient conditions for the existence and uniqueness of the optimal control . The form of the optimal control was also established as:

$$u^* = Sgn\,[c^T \sum_{i=0}^{N} S(t_0,s) B_i(s+h_i)]$$

It is concluded that, if the given system is normal, optimal control is unique and Bang Bang. It was also shown that, if u^* is the optimal control with t^* the minimum time, then, $u(t^*) \in \partial\Re(t^*, t_0)$, that is, the boundary of the reachable set.

Optimal controls as emphasized earlier literally means controlling a system in the 'best conceivable way'. This has been observed in most controlled linear processes of certain types. Exploits are now directed at nonlinear systems too of many types where unavoidable nonlinearities in systems affect the evolution of the system in a direct manner. The findings of this thesis has x-rayed and resolved such nonlinearities in neutral control systems. More interestingly is the fact that the neutral control system is shown to be not only relatively controllable but also optimally controllable.

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