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Analysis of Simply Supported Elastic Beam Resting on A Variable Pasternak Foundation Subjected to Moving Load With Damping Term

ATITI, S. EHIDIAME[1], MOMOH, S. OMEIZA (PhD)[2], GYEGWE, M. JESSICA (PhD)[3], EDOGBANYA, O. HELEN (PhD)[4]

[1, 2, 3, 4]*Department of Mathematics, Federal University Lokoja, Kogi State.*

Abstract- This paper investigated the dynamic response of a simply supported uniform Bernoulli–Euler beam resting on a variable Pasternak foundation under the action of a concentrated moving load with a damping term. To solve the fourth order partial differential equation that describes the beam's behaviour, the Dirac delta function was expressed as a Fourier cosine series and the Galerkin method, the Struble's asymptotic technique and the Laplace transform method were used. Thereafter, the deflection profiles of the beam were determined for several values of the parameters. The findings revealed that, the moving force problem is structurally unsafe to approximate the moving mass problem in the design of the dynamical system. Furthermore, as the values of the damping term, axial force, foundation modulus and shear modulus increases, the deflection profiles of the beam decreases with that of the damping term showing a far higher effect on the beam's deflection. This implies that the beam's functional performance are ensured when the values of each parameter is increased.

Index Terms- Bernoulli-Euler Beam, Damping Term, Simply Support Condition, Variable Pasternak Foundation

I. INTRODUCTION

The study of dynamic responses of structural members such as beams resting on elastic foundation subjected to moving loads such as train and vehicles, is important by virtue of the relevance it has on the design and construction engineering, especially with bridges, rails, roadways and airport runways. These moving loads were quickly reported, to have a great effect on dynamic stresses in such structures, which cause them to vibrate intensively, especially at high velocities (Willis, *et al.*, 1851) since moving load induces larger deflections and stresses on the structure on which it moves than does an equivalent static load. In most cases, these vibrations create dynamic stresses and strains which can cause corrosion between contacting elements and lead to the failure of such structural members. Thus, vibration of structures has continued to attract the interest of engineers, physicists and applied mathematicians over the years due to the devastating effects.

One of the earliest works in available literature on the dynamic response of a simply supported beam is that of Krylo (1905), who investigated the dynamic response of a simply supported beam, traversed by a constant force moving at a uniform speed. His results were obtained by using the method of expansion of eigen-functions. In a related study, Timoshenko (1921) used energy methods to obtain solutions in series form for simply supported finite beam on an elastic foundation subjected to time dependent point loads moving with uniform velocities across the beam. Inglis (1934), limited his discussions to analyzing transverse oscillations induced by a uniform. He assumed that the beam is simply supported. Much later, Oni and Ayankop-Andi (2017) investigated the problem of a simply supported non-uniform Rayleigh beam under travelling distributed loads. Analytical and numerical solutions showed that, resonance was reached earlier in the moving distributed mass system than in the moving distributed force system. Still considering distributed loads, Onyia and Kwaghvihi (2020) investigated the dynamic response of a simply supported beam resting on elastic foundation subjected to a moving uniformly distributed load. The results showed that increase in the dynamic load length, moving at constant speed leads to higher deflections and bending moments of the beam, and decreases with increasing foundation modulus. Thereafter, Ogunbamike (2021) presented a complete information on how to use MA to derive the forced vibration responses of a simply thick beam subjected to harmonic moving loads. He investigated the dynamic response of the Timoshenko beam resting on an elastic foundation subjected to harmonic moving load using modal analysis (MA).

In the research works above on the dynamic responses of beams, the foundations have been on one parameter. In particular, they lie on the Winkler foundation model with foundation stiffness K. However, the Winkler foundation model has been criticized because it predicts discontinuities in the deflection of the surface of the foundation at the ends of a finite beam, which is a contradiction to observation made in practice. In an attempt to eliminate this shortcoming, an improved theory called a two-parameter foundation model was proposed by Pasternak (1954) for the analysis of the dynamic behavior of beams under moving loads. Pasternak improved this model by adding a shear spring to simulate the interactions between separate springs in the Winkler model. For this model, aside the foundation stiffness K, a second foundation parameter, the shear modulus G, enters the analysis.

The dynamic analysis of simply supported beam resting on two parameters foundation models under moving loads has been investigated. Ojih, *et al.* (2013) investigated the dynamic response of non-uniform Rayleigh beam resting on Pasternak foundation and subjected to concentrated loads travelling at varying velocity with simply supported boundary condition. The study showed that, we cannot guarantee safety for a design based on the moving force solution since resonance is reached earlier in the moving mass problem than in the moving force problem. By considering the inertia effect of the load also, Oni and Jimoh (2016) investigated the dynamic response to non-uniform simply supported prestressed Bernoulli Euler beam. It was found that as the parameters increases, the displacement responses of the beam decreases. Furthermore, the moving force solution is not an upper bound for an accurate solution of the moving mass problem. The beam rests on a Pasternak foundation and traversed by concentrated moving loads. By considering a distributed moving load, Awodola, *et al.* (2019) investigated the dynamic response to variable magnitude moving distributed masses of simply supported non-uniform Bernoulli–Euler beam resting on Pasternak elastic foundation. The displacement response for moving distributed force and moving distributed mass models for the dynamical problem are calculated for various time t and presented in plotted curves. Still on distributed loads, Jimoh and Ajoge (2020) investigated the dynamic analysis of non-uniform Bernoulli-Euler beam resting on bi-parametric foundations and traversed by constant magnitude moving distributed load with simply supported ends conditions. Damping term effect was incorporated into the model. The deflection of the beam under moving loads is calculated for several values of damping coefficient, shear modulus, axial force and foundation modulus. Others who used the two parameter foundations are Abbas, *et al.* (2021), Akhazhanov, *et al.* (2023), Awodola, *et al.* (2024), Sulaiman, *et al.* (2024).

Although all the researchers above achieved tremendous feat in the dynamic study of structures they considered, all the foundations were constant. However, this is not always the case in practice where such foundations may vary spatially. A variable foundation means that the stiffness of the foundation varies along the length of the beam. In the governing equation of a variable foundation, the foundation stiffness and the shear modulus vary. This renders the exact solution for the dynamical problem difficult to obtain as the governing partial differential equation now has variable coefficients. Interestingly, there are researchers who tackled problems with variable elastic foundations that got wonderful results. Some of the researchers are Oni and Awodola (2003) who proposed an elegant method based on the generalized Galerkin's method and Struble's asymptotic technique to assess the vibration under a moving concentrated load of a simply supported non-uniform Rayleigh beam on variable elastic foundation. Later, Abdelghany, *et al.* (2015) investigated the response of non-uniform Euler– Bernoulli simply supported beam which is subjected to moving load. The beam is rested on a nonlinear viscoelastic foundation. The influences of variations of the traveling velocity and the effect of increase in the magnitude of the moving load on the dynamic response were studied. In addition, Ogunyebi and Adedowole (2017) in a study obtained analytical solutions to the non-homogeneous fourth order partial

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differential equation governing simply supported Rayleigh Beam with an accelerating distributed mass on elastic foundation. The dynamic effects of vital parameters such as elastic foundation, rotatory inertia correction factor, axial force, distance and load parameters were obtained. Saurabh (2021) analysed an axially functionally graded Euler-Bernoulli beam resting on variable Pasternak foundation. The simply supported was used in the analysis. The problem was formulated using Rayleigh-Ritz method and governing equations are derived with the help of Hamilton's principle. Other researchers who considered cases of variable foundation are Kacar, *et al.* (2011), Mirzabeigy and Madoliat (2016), Zhang, *et al.* (2016), Ogunyebi (2017) and Yas, *et al*. (2017).

From the works reported so far and to the authors' knowledge from available literature, cases where the damping term is considered are fewer. In addition, where the damping term was considered, the inertia effect of the moving load was generally not considered. Specifically, the case of a simply supported uniform Bernoulli-Euler resting on a variable Pasternak foundation under the action of a concentrated moving load with damping term, where the initial effects of the load is incorporated in its analysis has not been considered. In this study therefore, this case is considered to provide valuable insights for designing and optimizing such dynamical systems. Thus, this paper investigated a simply supported uniform Bernoulli-Euler beam resting on a variable Pasternak foundation subjected to a concentrated moving load with damping term, when the inertia effect of the moving load is considered.

II. METHOD

2.1. Formulation of the Governing Equation

The governing equation for a uniform Bernoulli Euler beam resting on constant Pasternak foundation subjected to concentrated moving load with damping term is given as:

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} + \bar{m}\frac{\partial^2 W(x,t)}{\partial t^2} + \alpha\frac{\partial W(x,t)}{\partial t} - N\frac{\partial^2 W(x,t)}{\partial x^2} + KW(x,t) - G\frac{\partial^2 W(x,t)}{\partial x^2} = mg\delta(x-ct)\left[1-\frac{1}{g}\frac{d^2 W(x,t)}{dt^2}\right] \quad (1)$$

where x is the spatial coordinate, t is the time, $W(x,t)$ is the transverse displacement, E is the Young's modulus, I is the moment of inertia, EI is the flexural rigidity of the structure, $\bar{m}$ is the mass per unit length of the beam, α is the damping term coefficient, N is the axial force, K is the foundation modulus and G is the shear modulus, m and c are the mass and the speed of the moving load respectively, g is the acceleration due to gravity, $mg\delta(x-ct)$ is the continuous moving force acting on the beam.

Where $\frac{d^2}{dt^2}$ is a convective acceleration given by Fryba (1972) as

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x\partial t} + c^2\frac{\partial^2}{\partial x^2} \quad (2)$$

when the operator, $\frac{d^2}{dt^2}$ acts as the transverse deflection $W(x,t)$ of the beam, the first term in the right-hand side of the equation (11), measures the effect of the acceleration on the deflection, the second term measures the effect of complementary acceleration (coriohs force) and the third term measures the effect of the path curvature (centripetal force).

The boundary conditions of the structure are arbitrary while the initial condition is given as:

$$W(x,0) = 0 = \frac{\partial W(x,t)}{\partial t} \quad (3)$$

For the variable Pasternak foundation,

$$K(x)W(x,t) - \frac{\partial}{\partial x}\left[G(x)\frac{\partial W(x,t)}{\partial x}\right] = K(x)W(x,t) - G'(x)\frac{\partial W(x,t)}{\partial x} - G(x)\frac{\partial^2 W(x,t)}{\partial x^2} \quad (4)$$

Where $K(x)$ is the variable foundation stiffness, $G(x)$ is the variable shear modulus and they are given as:

$$K(x) = k_0\,(4x - 3x^2 + x^3) \quad (5)$$

$$G(x) = G_0\,(12 - 13x + 6x^2 - x^3) \quad (6)$$

$$G'(x) = G_0\,(-13 + 12x - 3x^2) \quad (7)$$

where

Using equations (2), (4), (5), (6) and (7) in equation (1), after simplifications, we obtain:

$$\frac{EI}{\bar{m}}\frac{\partial^4 W(x,t)}{\partial x^4}+\frac{\partial^2 W(x,t)}{\partial t^2}-\frac{N}{\bar{m}}\frac{\partial^2 W(x,t)}{\partial x^2}+\frac{k_0\,(4x-3x^2+x^3)}{\bar{m}}W(x,t)-\frac{G_0\,(-13+12x-3x^2)}{\bar{m}}\frac{\partial W(x,t)}{\partial x}-\frac{G_0}{\bar{m}}(12-13x+6x^2-x^3)\frac{\partial^2 W(x,t)}{\partial x^2}+\frac{\alpha}{\bar{m}}\frac{\partial W(x,t)}{\partial t}+\frac{m}{\bar{m}}\delta(x-ct)\left\{\frac{\partial^2 W(x,t)}{\partial t^2}+2c\frac{\partial^2 W(x,t)}{\partial x\partial t}+c^2\frac{\partial^2 W(x,t)}{\partial x^2}\right\}=\frac{mg}{\bar{m}}\delta(x-ct) \quad (8)$$

where

$\delta(x-ct)$ represents the dira delta function defined as

$$\delta(x-ct)=\begin{cases}0, x\neq ct\\ \infty, x=ct\end{cases} \quad (9)$$

with property

$$\int_0^L \delta(x-ct)f(x)dx=\begin{Bmatrix}0; ct<0\\ f(x); 0<ct<L\\ 0; ct>L\end{Bmatrix} \quad (10)$$

Equation (8) is the simplified governing equation for uniform Bernoulli Euler beam resting on variable Pasternak foundation subjected to concentrated moving load with damping term, when the initial effect of the moving load is put into consideration.

2.2 Solution of the Problem

For the variable foundation, the approximate method best suited for solving diverse problems in dynamics of structures generally referred to as Galerkin's method is used. It has a solution of the form:

$$W(x,t)=\sum_{i=0}^{n} Y_m(t)U_m(x) \quad (11)$$

is sought, where $U_m(x)$ is chosen as a suitable kernel of the Galerkin method in Eq. (8).

$$U_m(x)=\mathrm{Sin}\frac{\lambda_m x}{L}+A_m Cos\frac{\lambda_m x}{L}+B_m Sinh\frac{\lambda_m x}{L}+C_m Cosh\frac{\lambda_m x}{L} \quad (12)$$

is chosen such that the boundary conditions are satisfied. Where λ_m is the mode frequency. *Am*, *Bm*, *Cm* are constants which are obtained by substituting (12) into the appropriate boundary conditions.

Eqn (8) can be rewritten as;

$$H_1\frac{\partial^4 W(x,t)}{\partial x^4}+\frac{\partial^2 W(x,t)}{\partial t^2}-H_2\frac{\partial^2 W(x,t)}{\partial x^2}+H_3(4x-3x^2+x^3)W(x,t)$$

$$-H_4(-13+12x+3x^3)\frac{\partial W(x,t)}{\partial t}-H_4(12-13x+6x^2-x^3)\frac{\partial^2 W(x,t)}{\partial x^2}+H_5\frac{\partial W(x,t)}{\partial t}$$

$$-\frac{M}{\bar{m}}\delta(x-ct)\left[\frac{\partial^2 W(x,t)}{\partial t^2}+2c\frac{\partial^2 W(x,t)}{\partial x\partial t}+c^2\frac{\partial^2 W(x,t)}{\partial x^2}\right]=\frac{Mg}{\bar{m}}\delta(x-ct) \quad (13)$$

Where:

$$H_1=\frac{EI}{\bar{m}};\quad H_2=\frac{N}{\bar{m}};\quad H_3=\frac{K_0}{\bar{m}};\quad H_4=\frac{G_0}{\bar{m}};\quad H_5=\frac{\alpha}{\bar{m}}$$

Using equation (8) on (13), substitutions, simplifications, and using the generalized Galerkin method, we get;

$$\sum_{m=1}^{n}\{G_{A0}\ddot{Y}_m(t)+H_5 G_{A0}\dot{Y}_m(t)+[H_1G_{A4}-H_2G_{A2}+H_3(4x-3x^2+x^3)G_{A0}-H_4(-13+12x-3x^2)G_{A1}-H_4(12-13x+6x^2-x^3)G_{A2}\}Y_m(x)+\frac{M}{\bar{m}}[G_{fo}(t)\ddot{Y}_m(t)+2cG_{f1}\dot{Y}_m(t)+c^2G_{f2}(t)Y_m(x)]\frac{Mg}{\bar{m}}G_H(t) \quad (14)$$

Where:

$G_{A0}=\int_0^L U_m(x)U_k(x)\,dx$; $G_{A1}=\int_0^L U'_m(x)\,dx$; $G_{A2}=\int_0^L U''_m(x)U_k(x)\,dx$

$G_{A4}=\int_0^L U_m^{(iv)}(x)U_K(x)\,dx$;

$G_{f0}(t)=\int_0^L \delta(x-ct)\,G_{A0}$; $G_{f1}(t)=\int_0^L \delta(x-ct)G_{A1}=\frac{G_{A1}}{L}+\frac{2G_{B1}}{L}$; $G_{f2}(t)=\int_0^L \delta(x-ct)U_k(x)$; $G_H(t)=\int_0^L \delta(x-ct)U_k(x)$

Equation (14) is the transformed equation governing the problem. This non-homogeneous 2nd order ODE

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holds for all variants of the classical Boundary Conditions.

2.3 Case 1: Uniform Bernoulli Euler Beam Traversed by Concentrated Moving Force

Equation (14) can be rewritten as:

$$\sum_{m=1}^{n}\left[Q_A(m,k)\ddot{Y}_m(t) + Q_B(m,k)\dot{Y}_m(t) + Q_C(m,k)Y_m(t)\right]$$

$$+\varepsilon_0(Q_D(m,k)\ddot{Y}_m(t) + Q_E(m,k)\dot{Y}_m(t) + Q_F(m,k)Y_m(t)]$$

$$= \left[\frac{Mg}{\bar{m}}\right] Q_G(m,k) \quad (15)$$

Where Q_A(m,k), Q_B(m,k), Q_C(m,k), Q_D(m,k), Q_E(m,k), Q_F(m,k), $Q_G(m,k)$ are solutions of respective integrals.

For this case, we assume that the inertia effect of the moving mass is negligible and set $\varepsilon_0 = 0$ in equation (15), only the force effect of the moving concentrated load is considered.

Eqn (15) reduces to:

$$\ddot{Y}_m(t) + \frac{Q_B(m,k)}{Q_A(m,k)}\dot{Y}_m(t) + \frac{Q_C(m,k)}{Q_A(m,k)}Y_m(t) = \frac{Mg}{\bar{m}} \cdot \frac{Q_G(m,k)}{Q_A(m,k)} \quad (16)$$

This represents the classical case of a moving concentrated force problem associated with the dynamical system. Evidently, an exact analytical solution to this equation is not possible, though the equation yields readily to numerical solution, an analytical methods is desirable as the solution so obtained often shed light on the vital information about the vibrating system. To this end, a modification of the asymptotic method due to Struble, often used in treating weakly homogeneous and non-homogeneous non-linear oscillatory systems shall be used to solve the problem. Consequently equation (16) can be rearranged to take the form:

$$\ddot{Y}_m(t) + U^2{}_{mf}\,(t)(t)\,\dot{Y}_m(t) + V^2{}_{mf}(t)Y_m(t) = \frac{Mg}{\bar{m}}W^2{}_{mf}(t) \quad (17)$$

Where

$$U^2{}_{mf}\,(t) = \frac{Q_B(m,k)}{Q_A(m,k)}, V^2{}_{mf}(t) = \frac{Q_C(m,k)}{Q_A(m,k)}, W^2{}_{mf}(t) = \frac{Q_G(m,k)}{Q_A(m,k)}$$

To obtain the solution to (17), it is subjected to Laplace transform defined as

$$F(s) = \int_0^\infty e^{-st} f(t)dt\,; s > 0 \quad (18)$$

where s is the Laplace parameter in conjunction with initial conditions defined on $Y_m(t)$, one obtains:

$$\mathcal{L}\left[\ddot{Y}_m(t)\right] + U^2{}_{mf}\,(t)\mathcal{L}\left[\dot{Y}_m(t)\right] + V^2{}_{mf}(t)\mathcal{L}[Y_m(t)] = \mathcal{L}\left[\frac{Mg}{\bar{m}}W^2{}_{mf}(t)\right] \quad (19)$$

By evaluating 19, finding the Laplace inversion, making use of the convolution integral defined as

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du \quad (20)$$

we obtain

$$Y_m(t) = R_A\left[F_f(s) * G_{1f}(s) - F_f(s) * G_{2f}(s)\right] \quad (21)$$

where

$$R_A = \frac{R}{\alpha_1 - \alpha_2}$$

$$F_f\,(t) = \mathcal{L}^{-1}[F_f(s) = F_A + F_B\cos\theta_n t \quad (22)$$

$$g_{1f}(t) = \mathcal{L}^{-1}[G_{1f}(s) = e^{-\alpha_2 t} \quad (23)$$

$$g_{2f}(t) = \mathcal{L}^{-1}[G_{2f}(s) = e^{-\alpha_1 t} \quad (24)$$

where

$$F_A = \frac{1}{L}[I_a + A_m I_b + B_m I_c + C_m I_d];$$

$$I_a = \frac{L(1-\cos\lambda_m)}{\lambda_m};\quad I_b = \frac{L\sin\lambda_m}{\lambda_k};\quad I_c = L\frac{(-1+\cosh\lambda_m)}{\lambda_m}$$

$$I_d = \frac{L\sinh\lambda_m}{\lambda m};$$

$$F_B = \frac{2}{L}[F_1 + A_m F_2 + B_m F_3 + C_m F_4]\cos\left(\frac{n\pi ct}{L}\right);$$

$F_B = F_{B1} \cos\left(\frac{n\pi ct}{L}\right)$;

$F_{B1} = \frac{2}{L}[F_1 + A_m F_2 + B_m F_3 + C_m F_4]$;

$F_1 = \frac{L(\lambda_k - \lambda_k \cos\lambda_k \cos n\pi - n\pi \sin\lambda_k \sin n\pi)}{{\lambda_k}^2 - n^2\pi^2}$;

$F_2 = \frac{L(\lambda_k \cos n\pi \sin\lambda_k - n\pi \cos\lambda_k \sin n\pi)}{{\lambda_k}^2 - n^2\pi^2}$;

$F_3 = \frac{L(-\lambda_k + \lambda_k \cos n\pi \cosh\lambda_k + n\pi \sin n\pi \sinh\lambda_k)}{{\lambda_k}^2 + n^2\pi^2}$

$F_4 = \frac{L(n\pi \cosh\lambda_k \sin n\pi + \lambda_k \cos n\pi \sinh\lambda_k}{{\lambda_k}^2 + n^2\pi^2}$;

$\alpha_1 = -\frac{1}{2}\left[{U^2}_{mf} + \sqrt{{U^4}_{mf} - 4{V^2}_{mf}}\right]$;

$\alpha_2 = -\frac{1}{2}\left[{U^2}_{mf} - \sqrt{{U^4}_{mf} - 4{V^2}_{mf}}\right]$

Applying (20), substitutions equations and after some simplification, with substitution into equation (11), we obtain

$$W(x,t) = \sum_{m=1}^{n} R_A \left[F_A\left(\frac{1-e^{-\alpha_2 t}}{\alpha_2} - \frac{1-e^{-\alpha_1 t}}{\alpha_1}\right)\right]$$

$$+F_{B1}\left(\frac{\theta_n}{{\theta^2}_n + {\alpha_2}^2}\left[\sin\theta_n t + \frac{\alpha_2}{\theta_n}(\cos\theta_n t - e^{-\alpha_2 t})\right] - \frac{\theta_n}{{\theta^2}_n + {\alpha_1}^2}\left[\sin\theta_n t + \frac{\alpha_1}{\theta_n}(\cos\theta_n t - e^{-\alpha_1 t})\right]\right) \quad (25)$$

Equation (25) above represents the transverse displacement response of the uniform Bernoulli-Euler beam under the action of moving concentrated force resting on a variable Pasternak foundation.

2.4 Case Two: Uniform Bernoulli Euler Beam Traversed by Concentrated Moving Mass

Retaining ϵ_0 in equation (14) and simplification, we have

$$\ddot{Y}_m(t) + {U^2}_{mf}(t)(t)\dot{Y}_m(t) + {V^2}_{mf}(t)Y_m(t) + \varepsilon_0[H_1(m,k)\ddot{Y}_m(t) + H_2(m,k)\dot{Y}_m(t) + H_3(m,k)Y_m(t)] = \frac{Mg}{\bar{m}\,Q_A(m,k)}Q_G(m,k) \quad (26)$$

Simplifying 26, we get

$$\ddot{Y}_m(t) + \frac{[{U^2}_{mf} + \varepsilon_0 H_2]}{1+\varepsilon_0 H_1}\dot{Y}_m(t) + \frac{[{V^2}_{mf} + \varepsilon_0 H_3(t)]}{1+\varepsilon_0 H_1}Y_m(t) = \frac{R}{1+\varepsilon_0 H_1}Q_G(m,k) \quad (27)$$

Where

$${U^2}_{mf}(t) = \frac{Q_B(m,k)}{Q_A(m,k)}\ ;\ {V^2}_{mf}(t)\frac{Q_C(m,k)}{Q_A(m,k)};$$

$$H_1(m,k) = \frac{Q_D(m,k)}{Q_A(m,k)};$$

$H_2(m,k) = \frac{Q_E(m,k)}{Q_A(m,k)}$; $H_3(m,k) = \frac{Q_F(m,k)}{Q_A(m,k)}$; $\varepsilon_0 = \frac{M}{\bar{m}}$;

$R = \frac{Mg}{mQ_A(m,k)}$; $R_1(t) = 1 + +\varepsilon_0 H_1$

By considering parameter $\epsilon < 1$, for any arbitrary mass ratio ε_0 and by simplifying,

$$\varepsilon_0 = \epsilon(1-\epsilon)^{-1} \quad (28)$$

Expanding (28) using the theory of binomial expansion of integers, using 27 and after substitution and simplification, we obtain

$$\ddot{Y}_m(t) + ({U^2}_{mf} + \epsilon H_2)(1 + \epsilon H_1)\dot{Y}_m(t) + ({V^2}_{mf} + \epsilon H_3)(1 + \epsilon H_1)Y_m(t) =$$

$$(1 + \epsilon H_1)RQ_g(m,k) \quad (29)$$

Expanding (29) and considering only up to $o(\epsilon)$, we obtain

$$\ddot{Y}_m(t) + [{U^2}_{mf} + \epsilon(H_1{U^2}_{mf} + H_2]\dot{Y}_m(t) + [{V^2}_{mf} + \epsilon(H_1{V^2}_{mf} + \epsilon H_3)]Y_m(t) = (1 + \epsilon H_1)RQ_g(m,k) \quad (30)$$

Setting $\epsilon = 0$, in equation (30), a situation corresponding to the case in which the inertial effect is regarded as negligible is obtained, then the solution can be written as:

$$Y_{mf}(m,t) = \beta_{mf}\cos(\alpha_{mf}t - w_{mf}) \quad (31)$$

Where β_{mf}, α_{mf}, ω_{mf} are constants

Since $\epsilon < 1$, an asymptotic expansion of the homogenous part of the expression (30) can be written as

$$Y_m(t) = \phi(m,t)\ \text{Cos}[V_{m,f}t - \Omega(m,t)] + \epsilon\, Y_{1m}(t) + 0(\epsilon^2) \qquad (32)$$

Where $\phi(m,t)$, and $\Omega(m,t)$ are slowly tone varying functions or equivalently

In view of (32) and after some simplification, neglecting $O(\epsilon^2)$ parts, we have

$$\dot{Y}_m(t) = \dot{\phi}(m,t)\,\text{Cos}(V_{mf}t - \Omega(m,t) - \phi(m,t)V_{m,f}Sin\ t - \Omega(m,t) + \phi(m,t)\dot{\Omega}(m,t)$$

$$Sin(V_{mf}t - \Omega(m,t) + \epsilon\dot{Y}im(t) \qquad (33)$$

and

$$\ddot{Y}_m(t) = -\dot{2}\phi(m,t)V_{mf}\ \text{Sin}(V_{mf}t - \Omega(m,t) - 2\phi(m,t)V_{m,f} - \dot{\Omega}(m,t)\ Cos(V_{mf}t - \Omega(m,t) - \phi(m,t)V_{mf}^2 Cos(V_{mf}t - \Omega(m,t) + \epsilon\ddot{Y}im(t) \qquad (34)$$

Substituting (32), (33) and (34) into the homogenous part of (30) and considering only $O(\epsilon)$, we have

$$-2\dot{\phi}(m,t)V_{mf}Sin\left(V_{mf}t - \Omega(m,t)\right) + 2\phi(m,t)\ V_m f\dot{\Omega}\,(mt)\ \text{Cos}(V_{mf}t - \Omega\,(m,t))\text{-}\ \phi\,(m,t)V_{mf}^2\cos\,(V_{mf}t - \Omega\,(m,t) + \epsilon\ddot{Y}_{1m}(t)\text{-}\ \phi\,(m,t)U_{mf}^2\ Sin\ (V_{mf}t - \Omega\,(m,t) + \epsilon U_{mf}^2\dot{Y_{\imath m}}(t)\ \text{-}\ \epsilon Z_1\phi\,(m,t)V_{mf}Sin(V_{mf}t - \Omega\,(m,t)) + \epsilon Z_2\phi(m,t)Cos\ (V_{mf}t - \Omega\,(m,t) = 0 \qquad (35)$$

Where

$Z_1 = H_1U_{mf}^2 + H_2$, $Z_2 = H_1V^2_{mf} + H_3$

In order to obtain the modified frequency, we extract only the variational parts of equation (35) that describe the behaviour of $\phi(m,t)$ and $\Omega(m,t)$ during the motion of this mass. That is, neglecting terms without $\text{Sin}(V_{mf}t - \Omega\,(m,t)$ and $\text{Cos}(V_{mf}t - \Omega\,(m,t)$

Hence, we have

$$[-2\phi(m,t)\ V_{mf} - \phi(m,t)U^2V_{mf} - \epsilon Z_1\ \phi(m,t)V_{mf}]\ sin(V_{mf}t - \Omega(m,t) + [2\phi(m,t)\ V_{mf}\dot{\Omega}(m,t)\ \ - \phi(m,t)V^2{}_{mf} + V^2{}_{mf}\phi(m,t) +\ \epsilon Z_2\ \phi(m,t)]\ Cos(V_{mf}t - \Omega(m,t) \qquad (36)$$

Equating the coefficient of Cos $(V_{n,f}t - \Omega(m,t))$ and Sin $(V_{n,f}t - \Omega(m,t))$ to zero, we obtain:

$$\left[2\dot{\phi}(m,t) + \epsilon Z_1 +\ U^2{}_{mf}\ \phi(m,t)\right]V_{mf} = 0 \qquad (37)$$

And

$$[2V_{mf}\dot{\Omega}(m,t) + \epsilon Z_2\phi(m,t) = 0 \qquad (38)$$

From (37) and 38, we obtain

$$\phi(m,t) = \text{A}\ e^{-\left[\frac{\epsilon Z_1 + U^2_{mf}}{2}\right]t} \qquad (39)$$

$$\therefore \Omega(m,t) = -\frac{\epsilon Z_2}{2V_{mf}} + C_m \qquad (40)$$

Where C_m is a constant of integration

Given that:

$$Y_m(t) = \phi(C_{m,}t)\ \text{Cos}\ \ (V_{mf}t - \Omega(m,t)) \qquad (41)$$

Substituting (39) and (40) into (41) and after substitution and some simplification, we get

$$V^2{}_{mm} = \ V_{mf}\ (1 + \frac{\epsilon Z_2}{2V_{mf}}) \qquad (42)$$

Equation (42) is called the modified natural frequency, representing the frequency of the free system due to presence of the moving force. Thus, to solve the non-homogenous equation (17), the differential operator in $Y_m(t)$ is replaced by V_{mm}^2 Hence, one obtains

$$\ddot{Y}_m(t) + U^2{}_{mf}(t)\ \dot{Y}_m(t) +\ V^2{}_{mm}(t)\ Y_m(t) = \ RQ_g(m,k) \qquad (43)$$

After some substitution and using Laplace transform method and after some substitution into equation (11)

$$W(x,t) = \sum_{m=1}^{n} R_A \ \{F_A\left[\frac{1-e^{-r_2 t}}{r_2} - \frac{1-e^{-r_1 t}}{r_1}\right] + $$
$$+ F_{B1}\left[\frac{\theta_n}{{\theta_n}^2 + {r_2}^2}\left[Sin\theta_n t + \frac{r_2}{\theta_n}\left(\text{Cos}\theta_n t - e^{-r_2 t}\right)\right] - \right.$$
$$\frac{\theta_n}{{\theta_n}^2 + {r_1}^2}\left[Sin\theta_n t + \frac{r_1}{\theta_n}\left(Cos\theta_n t - e^{-r_1 t}\right]\right\} U_m(x) \quad (44)$$

$$r_1 = -\frac{1}{2}\left[U^2{}_{mf} + \sqrt{U^4{}_{mf} - 4V^2{}_{mm}}\right];$$

$$r_2 = -\frac{1}{2}\left[-U^2{}_{mf} + \sqrt{U^4{}_{mf} - 4V_{2mm}}\right]$$

Equation (44) above represents the transvers displacement response of the uniform Bernoulli-Euler beam resting on a variable Pasternak foundation under the action of a moving mass.

III. APPLICATION OF THE SIMPLY SUPPORTED BOUNDARY CONDITION

The simply supported boundary conditions are

$$W(0,t) = W(L,t) = 0, \ \frac{\partial^2 w(0,t)}{\partial x^2} = 0 = \frac{\partial^2 w(L,t)}{\partial x^2} \quad (45)$$

Consequently, for the normal modes

$$U_m(0) = U_m(L) = 0, \frac{\partial^2 {U_m}^{(0)}}{\partial x^2} = 0 = \frac{\partial^2 {U_m}^{(L)}}{\partial x^2} \quad (46)$$

It is also clear that

$$U_k(0) = U_k(L) = 0, \ \frac{\partial^2 {U_k}^{(o)}}{\partial x^2} = 0 = \frac{\partial^2 U_k(L)}{\partial x^2} \quad (47)$$

Using the above Boundary Condition in equation (12), results to

$$A_m = B_m = C_m = 0$$

and$A_k = B_k = C_k = 0$

and the frequency equation is obtained as:

$$Sin\lambda_m = 0 \ ,$$

Thus, $\lambda_m = m\pi$

Similarity ,$Sin\lambda_k = 0$ then $\lambda_m = k\pi$

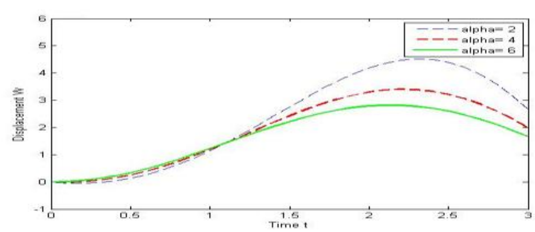
Substituting the $A_m = B_m = C_m = 0$ into equation (25) and (44), we obtain the equation that describes the dynamic response of a simply supported uniform Bernoulli Euler beam resting on a variable Pasternak foundation subjected to a concentrated moving force and moving mass respectively with a damping term.

IV. NUMERICAL ANALYSIS AND DISCUSSION OF RESULTS

4.1. Numerical Analysis

Numerical analysis for both moving concentrated force and moving concentrated mass problems for the simply supported uniform Bernoulli-Euler beam were simulated using MATLAB. This was done by considering a homogenous beam of modulus of elasticity $E = 2.02 \times 10^{11}\ N/m^2$, the moment of inertia $I = 2.87698 \times 10^{-3} m^4$, the beam span $L = 100m$, the mass per unit length of the beam $\bar{m} = 2758.291\ Kg/m$.

With respect to the four parameters, the values of axial force N was varied between $0N\ and\ 20000000N$, the values of the shear modulus G varies between $0\ N/m^3 and\ 50000000\ N/m^3$, the values of k varies between $0\ N/m^3 and\ 50000000\ N/m^3$ and the values of the damping coefficient α varied between 0 and 6. The results were shown in graphs for the simply supported boundary condition considered for varied and fixed values of the four parameters.



$Figure$ 1: Transverse displacement of the beam for various values of damping coefficient and fixed values of other parameters traversed by a moving concentrated force

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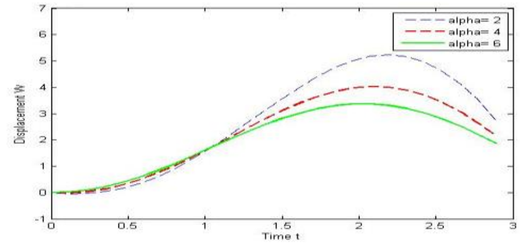
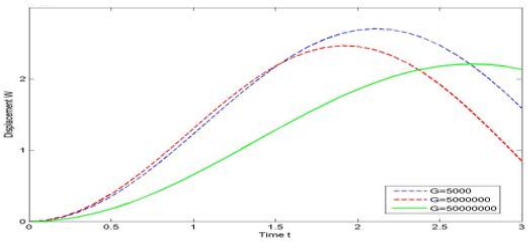
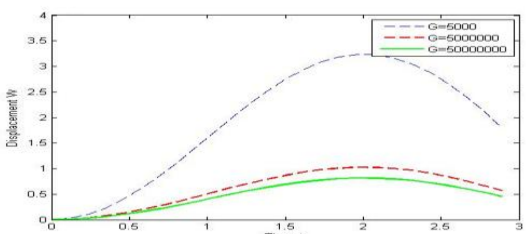


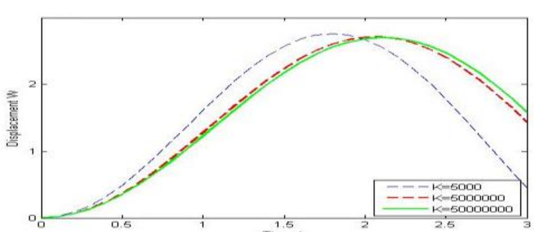
$Figure$ 2: Transverse displacement of the beam for various values of damping coefficient and fixed values of other parameters traversed by a moving concentrated mass



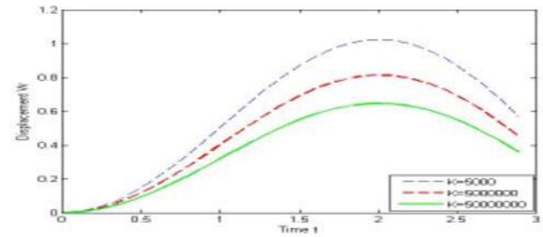
$Figure$ 3: Transverse displacement of the beam for various values of shear modulus and fixed values of other parameters traversed by a moving concentrated force



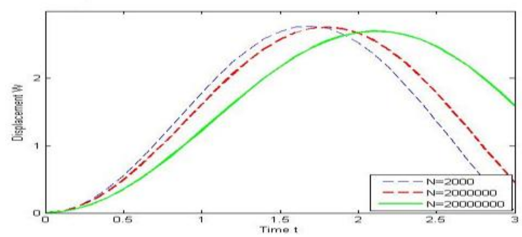
$Figure$ 4: Transverse displacement of the beam for various values of shear modulus and fixed values of other parameters traversed by a moving concentrated mass



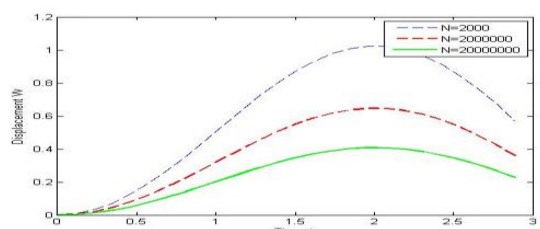
$Figure$ 5: Transverse displacement of the beam for various values of foundation modulus and fixed values of other parameters traversed by a moving concentrated force



$Figure$ 6: Transverse displacement of the beam for various values of foundation modulus and fixed values of other parameters traversed by a moving concentrated mass



$Figure$ 7: Transverse displacement of the beam for various values of axial force and fixed values of other parameters traversed by a moving concentrated force



$Figure$ 8: Transverse displacement of the beam for various values of axial force and fixed values of other parameters traversed by a moving concentrated mass

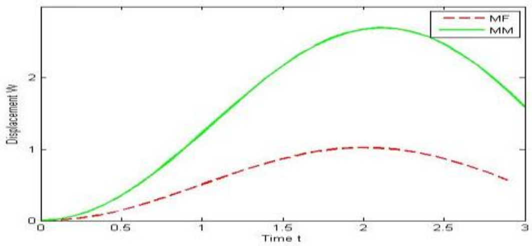


Figure 9: Comparison of the transverse displacement of the beam traversed by moving concentrated force and moving concentrated mass.

4.2. Discussion

The transverse displacements of the simply supported uniform Bernoulli-Euler beam resting on a variable Pasternak foundation under the action of moving load were presented in the figures above for values of the parameters. From the findings, an increase in the values of the damping term, shear modulus, foundation modulus and axial force reduced the deflection profile of simply supported uniform Bernoulli-Euler beam. In effect, their increase guaranteed a prolonged beam's life. In addition, the damping coefficient had a more noticeable effect on the response amplitude of the beam when compared with other parameters. This suggests that, the introduction of the damping term will ensure safety more than other parameters. Furthermore, figure 9 shows that the moving mass problem had a higher deflection profile than the moving force problem for the dynamical system. This implies that, the moving force problem is safer than that of the moving mass. However, it cannot be used as a safe approximation of the moving mass problem.

CONCLUSION

The study showed that increases in the values of the parameters decreased the transverse displacement of the simply supported beam for both moving concentrated force and moving concentrated mass cases. Thus, there increase ensured the beam's safety and stability. In addition, small changes in the values of the damping coefficient lead to more noticeable effects on the beam's displacement. Furthermore, the moving concentrated force problem was not a safe approximation for the moving concentrated mass problem. This work contributes to the existing knowledge on beam behavior on a simply support condition, by providing valuable insight on its displacement when it rests on a variable foundation. Its finding has implications for engineers in the design of such dynamical system. That is, increasing the values of these parameters can ensure the safety add longevity of the beam. Further research should explore cases of other classical boundary conditions. The results herein, were found to agree with those in literature.

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