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Maximum Principle and Its Application to Elliptic Partial Differential Equations

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I. INTRODUCTION

The most important tool that investigates most properties of solutions and equations of second order elliptic and parabolic type is the maximum principle. This has types, and all these types enable us to obtain valuable information about the properties of solutions and perhaps the equations themselves. Originating from the classical theory of harmonic functions, it has been generalized and extended to a wide range of elliptic partial differential equations and plays a critical role in proving uniqueness, comparism theorems, a priori estimates, and regularity results.

II. PRELIMINARIES

We now define some concepts used in this paper.

Partial Differential Equations

A kth-order partial differential equation is an expression of the form

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0 \quad (x \in \Omega) \qquad (1)$$

where

$$u : \Omega \rightarrow R$$

is the unknown.

A general linear second order partial differential equation with constant coefficient in **R** is of the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g \text{ in } \Omega \quad (2)$$

Elliptic Partial Differential Equation

An elliptic partial differential equation (2) is an equation in which the DISCRIMINANT is less than zero(that is $b^2 - 4ac < 0$)

Laplace Equation

The laplace equation characterizes a large group of physical problems that are independent of the time, and for this reason they are usually called steady state problems. An example is $\Delta u = o$ i.e. $\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial t^2} = 0$.

Maximum Principle

The maximum principle is a fundamental concept in the theory of partial differential equations, particularly for harmonic functions governed by the Laplace equation. It provides essential insights into the behavior of solutions within a domain.

There are two types of maximum principle

1. Weak Maximum Principle
2. Strong Maximum Principle

Weak Maximum Principle

If u satisfies a uniformly elliptic partial differential equation in a bounded domain and is continuous up to the boundary, then the maximum of u in the closure of the domain is achieved on the boundary.

Strong Maximum Principle

If u achieves an interior maximum and the equation is non-trivial, then u must be constant.

ELLIPTIC OPERATOR

We call the operator

$P = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$, $a_{ij} = a_{ji}$ (3)

(i, j = 1, . . . , n), elliptic at $x = (x_1, x_2, \ldots, x_n)$ if and only if there is a positive constant μ(x) such that

$a_{ij}(x)\xi_i\xi_j \geq \mu(x)\xi_i\xi_i$ (4)

for any vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. The operator P is said to be elliptic in a domain Ω if it is elliptic at each point of Ω, and it is uniformly elliptic if (4) holds for each point of Ω and if there is a positive constant μ_0 such that $\mu(x) \geq \mu_0$ for all x in Ω

III. LITERATURE REVIEW

The maximum principle is a generalization of the elementary fact of calculus that any function which satisfies the inequality $f'' > 0$ on an interval [a, b] attains its maximum value at a or b.

In general, function which satisfy elliptic inequalities on a domain **Ω** in n-dimensional Euclidean space take their maxima on the boundary of **Ω**. This is the simplest form of the maximum principle. [2]

Maximum principle for solutions of second order elliptic equations have been used in the mathematics literature since the early nineteenth century.

Given that u is a solution of

$$\Delta u + f(u) = 0 \text{ in } D$$
$$u = 0 \text{ on } \partial D$$

It turns out that the function P attains its maximum at a point where ∇u = 0 if ∂D is convex. This result was first found by Payne and Stakgold (1972), and it marked the beginning of a series of papers that were all concerned with various generalizations and applications of maximum principles for such a function P associated with the solution of some boundary value problems. These maximum principle proved to be a very useful tool in deriving all kinds of a priori bounds. In addition, most of the bounds found by these methods have the nice feature of being optimal in some sense. The book of Protter and Weinberger (1967) presents maximum principle for elliptic problems. [6]

Strong Maximum Principle [7]

The first classical maximum principle is accrued to E. Hoph, together with an extended commentary and discussion of Hopf's paper. The comparism technique invented by Hopf to prove this principle, has since become a main mathematical tool for the study of second order elliptic partial differential equations and has generated an enormous number of important applications.

The strong maximum principle of Eberhard Hopf is a classical and foundational result of the theory of second order elliptic partial differential equations. It goes back to the maximum principle for harmonic functions, already known to Guass in 1839 on the basis of the mean value theorem. It also extends to maximum principle for singular quasilinear elliptic differential inequalities; a theory initiated particularly by Vásquez and Diaz in the 1980's, but with earlier intimations in the work of Benilan, Brezis and Crandall.

Patricia Pucci and James Serrin provided a clear explanation of the strong maximum principle from its beginnings, showed its relation with and differences from the classical theory of Hopf and developed the features of these ideas in rather unexpected byways. They emphasized and maintained the nonlinear nature of the operators involved, in contrast to the naive view sometimes expressed that Hopf's original results applies principally to linear operators.

They considered in the first instance the strong maximum principle and the compact support principle for quasilinear elliptic differential inequalities, under generally weak asumptions on the quasilinear operators in question, in the canonical divergence structure

$$\text{div}\,\{(|Du|)Du\} - f(u) \leq 0,\ u \geq 0 \qquad (5)$$

and $\text{div}\,\{(|Du|)Du\} - f(u) \geq 0,\ u \geq 0,$

in a domain (connected open set) Ω in R^n, $n \geq 2$. Here Δu denotes the vector gradient of the given function $u = u(x)$, $x \in R^n$. They assumed throughout their paper, that unless otherwise stated explicitly, the following conditions on the operator $A = A(\rho)$ and the nonlinearity $f = f(u)$,

(1) $A \in C(0, \infty)$

(2) $\rho \mapsto \rho A(\rho)$ is strictly increasing in $(0, \infty)$ and $\rho A(\rho) \to 0$ as $\rho \to 0$;

(3) $f \in C[0, \infty)$,

(4) $f(0) = 0$ and f is non-decreasing on some interval $(0, \delta)$, $\delta > 0$.

By the strong maximum principle for (5), we mean the statement that if u is a classical solution of (5) with $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$ in Ω.

In order for the strong maximum principle to hold for (4), it is necessary and sufficient either that $f(s) \equiv 0$ for $s \in [0, \mu)$, $\mu > 0$, or that $f(s) > 0$ for $s \in (0, \delta)$ and

$$\int_0^{\delta} \frac{ds}{H^{-1}(F(s))} = \infty \qquad (6)$$

The necessity of (6) for the case of the Laplace equation is accrued to Benilan, Brezis and Crandall, while for the p-Laplacisn, it is accrued to Vázquez.

The Hopf Maximum Principle [7]

Gauss in 1839 had the knowledge of the maximum principle for harmonic and subharmonic functions on the basis of the mean value theorem; an extension to elliptic inequalities however remained open until the twentieth century. Bernstein (1904), Picard (1905), Lichtenstein (1912, 1924) then obtained various results by difficult means, as well as use of regularity conditions for the coefficients of the highest order terms.

IV. MAIN RESULT

Theorem: Suppose u satisfies the inequality

$$Lu = a_{ij}(x)u_{,ij} + b_i(x)u_{ij} \geq 0$$

in some finite domain $D \subset E^n$, and the coefficients of L are bounded in Ω. Then u cannot assume its maximum at an interior point of Ω unless $u \equiv$ constant.

Proof (Protter and Weinberger (1967)) [10]

The strong maximum principle also has an extension to the higher-dimensional case.

One of the important methods in studying harmonic functions is the maximum principle. The maximum principle for a class of elliptic differential equations slightly more general than the Laplace equation is discussed in the next section.

The Weak Maximum Principle [10]

We assume Ω is a bounded domain in R^n.

Definition: Let u be a C^2-function in Ω. Then u is a subharmonic (or superharmonic) function in Ω if $\Delta u \geq$ (or $\leq$) 0 in Ω.

Theorem 2: Let Ω be a bounded domain in R^n and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be asubharmonic in Ω . Then u attains its maximum in $\overline{\Omega}$, that is,

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

Proof: If u has a local maximum at a point x_0 in Ω, then the Hessian matrix

$$\Delta u(x_0) = \text{tr}(\nabla^2 u(x_0)) \leq 0$$

Hence, in the special case that $\Delta u > 0$ in Ω, the maximum value of u in $\bar{\Omega}$ is attained only on $\partial\Omega$.

We then consider the general case and assume that Ω is contained in the ball B_R for some $R > 0$. For any $\varepsilon > 0$, we consider

$$u_\varepsilon(x) = u(x) - (R^2 - |x|^2).$$

Then

$\Delta u_\varepsilon = \Delta u + 2n\varepsilon \geq 2n\varepsilon > 0$ in Ω

The special case we just discussed implies u_ε attains its maximum only on $\partial\Omega$ and hence

$$\max_{\bar{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon,$$

Then

$$\max_{\bar{\Omega}} u_\varepsilon \leq \max_{\bar{\Omega}} u_\varepsilon + \varepsilon R^2 = \max_{\partial\Omega} u_\varepsilon + \varepsilon R^2$$

$$\leq \max_{\partial\Omega} u + \varepsilon R^2$$

Letting $\varepsilon \to 0$, gives us the desired result and using the fact that $\partial\Omega \subset \overline{\Omega}$.

A continuous function in $\bar{\Omega}$ always attains its maximum in $\bar{\Omega}$. Theorem 2 asserts that any subharmonic function continuous up to the boundary attains its maximum on the boundary $\partial\Omega$, but possibly also in Ω.

A class of elliptic equations slightly more general than the Laplace equation is discussed next. Let c and f be continuous functions in Ω. Considering

$$\Delta u + cu = f \text{ in } \Omega,$$

We require $u \in C^2(\Omega)$. The function c is referred to as the coefficient of the zeroth-order term. u is harmonic if $c = f = 0$.

A C^2 -function u is called a subsolution (or supersolution) if $\Delta u + cu \geq f$ (or $\Delta u + cu \leq f$). If $c = 0$ and $f = 0$, subsolutions (or supersolutions) are subharmonic (or superharmonic).

Weak Maximum Principle for Subsolutions

u^+ is the nonnegative part of u defined by $u^+ = max\{0, u\}$

Theorem 3 Let Ω be a bounded domain in R^n and c be a continuous function in Ω with $c \leq 0$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\Delta u + cu \geq 0 \text{ in } \Omega$$

Then u attains on $\partial\Omega$ its nonnegative maximum in $\bar{\Omega}$, that is,

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+.$$

Proof

We set $\Omega_+ = \{x \in \Omega; u(x) > 0\}$. If $\Omega_+ = \emptyset$, then $u \leq 0$ in Ω, so $u^+ \equiv 0$. If $\Omega_+ \neq \emptyset$, then

$\Delta u = \Delta u + cu - cu \geq -cu \geq 0$ in Ω_+.

Theorem 2 implies

$$\max_{\Omega_+} u = \max_{\partial\Omega_+} u = \max_{\partial\Omega} u^+$$

If $c \equiv 0$ in Ω, Theorem 3 reduces to Theorem 2 and we can draw conclusions about the maximum of u rather than its nonnegative maximum.

Theorem 3 holds for general elliptic differential equations. Let a_{ij}, b_i and c be continuous functions in Ω with $c \leq 0$. We assume

$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda\, |\xi|^2$ for any $x \in \Omega$ and any $\xi \in R^n$, $\mathbb{R}$

for some positive constant λ. This means that we have a uniform positive lower bound for the eigenvalues of (a_{ij}) in Ω. For $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $f \in (\Omega)$, cinsider the uniformly elliptic equation

$$Lu = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f \text{ in } \Omega$$

The corollary below is a simple consequence of Theorem 3.

Corollary 4. Let Ω be a bounded domain in R^n and c be a continuous function in Ω with $c \leq 0$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\Delta u + cu \geq 0 in \Omega,$$

$$u \leq 0 \text{ on } \partial\Omega$$

Then c ≤ 0 in Ω.

The Strong Maximum Principle

The weak maximum principle asserts that subsolutions of elliptic differental equations attain their non-negative maximum on the boundary of the coefficients of the zeroth-order term is non-positive. In fact, these subsolutions can attain their nonnegative maximum only on the boundary, unless they are constant. This is the strong maximum principle. To prove this , we need the following Hopf lemma.

For any C'-function u in $\overline{\Omega}$ that attains its maximum on ∂Ω, say at $x_0 \in \partial\Omega$, we have $\frac{\partial u}{\partial v}(x_0) \geq 0$. The Hopf lemma asserts that the normal derivative is in fact positive if u is a subsolution in Ω.

Lemma 5: Let B be an open ball in R^n with $x_0 \in \partial B$ and c be a continuous function in $\overline{B}$ with c ≤ 0. Suppose $u \in C^2(B) \cap C'(\overline{B})$ satisfies

$$\Delta u + cu \geq 0 \text{ in B.}$$

Assume $u(x) < u(x_0)$ for any x ∈ B and $u(x_0) \geq 0$. Then

$$\frac{\partial u}{\partial v}(x_0) > 0,$$

where v is the exterior unit normal to B at x_0.

Proof

We assume $B = B_R$ for some R>0. By the continuity of u up to ∂B_R, we have

$$u(x) \leq u(x_0) \text{ for any } x \in \overline{B}_R$$

For positive constants α and ε to be determined, we set

$$w(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2},$$

and

$$v(x) = u(x) - u(x_0) + \varepsilon\, w(x)$$

We compute w and v in $D = B_R \backslash \overline{B}_{\frac{R}{2}}$

We have that

$$\Delta w + cw = e^{-\alpha|x|^2}(4\alpha^2|x|^2 - 2n\alpha + e) - ce^{-\alpha R^2}$$

$$\geq e^{-\alpha|x|^2}(4\alpha^2|x|^2 - 2n\alpha + e)$$

Where we used c ≤ 0 in B_R. Since $\frac{R}{2} \leq |x| \leq R$ in D, we have

$$\Delta w + cw \geq e^{-\alpha|x|^2}(\alpha^2 R^2 - 2n\alpha + c) > 0 \text{ in D,}$$

if we choose ∝ sufficiently large. By c ≤ 0 and $u(x_0) \geq 0$, we obtain, for any ε > 0.

$$\Delta v + cv = \Delta u + cu + \varepsilon(\Delta w + cw) - cu(x_0) \geq 0 \text{ in D}$$

We discuss v on ∂D in two cases. First, on $\partial B_{\frac{R}{2}}$, we have $u - u(x_0) < 0$, and hence $u - u(x_0) < -\varepsilon$ for some ε > 0. Note that w < 1 on $\partial B_{\frac{R}{2}}$. Then for such an ε, we obtain v < 0 on $\partial B_{\frac{R}{2}}$. Second, for $x \in \partial B_R$, we have w(x) = 0 and $u(x) \leq u(x_0)$. Hence $v(x) \leq 0$ for any $x \in \partial B_R$ and $v(x_0) = 0$. Therefore, v ≤ 0 on ∂D.

In conclusion,

$$\Delta v + cv \geq 0 \text{ in } D$$

$$v \leq 0 \text{ on } \partial D$$

By the comparism principle, we have

v ≤ 0 in D.

In view of $v(x_0) = 0$, then v attains at x_0 its maximum in $\overline{D}$. Hence, we obtain

$$\frac{\partial v}{\partial v}(x_0) \geq 0,$$

and then

$$\frac{\partial v}{\partial v}(x_0) \geq -\varepsilon\frac{\partial w}{\partial v}(x_0) = 2\varepsilon\alpha R e^{-\alpha|x|^2} > 0$$

Remark 8: Lemma 7 still holds if we substitute for B any bounded C^1-domain which satisfies an interior sphere condition at $x_0 \in \partial\Omega$, namely, if there exists a ball B ⊂ Ω with $x_0 \in \partial B$. This is because such a ball B is tangent to ∂Ω at x_0.

APPLICATIONS OF MAXIMUM PRINCIPLE TO ELLIPTIC EQUATIONS [10]

Apriori Estimates

An important application of maximum principle is to prove the uniqueness of solutions of boundary value problems.

Also, more importantly is to derive a priori estimates. In derivation of a priori estimates, it is necessary to construct auxiliary functions. We use only the weak maximum principle for the discussion.

In this section, we derive a priori estimates for solutions to the Dirichlet problem and the Neumann problem.

Suppose Ω is a bounded and connected domain in R^n. Consider the operator L in Ω.

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We assume that a_{ij}, b_i, and c are continuous and $a_{ij}(x)\xi_i\xi_j \geq \lambda\, |\xi|^2$ for any $x \in \Omega$ and any $\xi \in R^n$, where λ is a positive number. We denote by Λ the sup norm of a_{ij} and b_i, that is

$$\max_{\Omega}|a_{ij}| + \max_{\Omega}|b_i| \leq \Lambda$$

Proposition 9: Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\begin{cases} Lu = f \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

For some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. If $c(x) \leq 0$, then the following holds

$$|u(x)| \leq \max_{\partial\Omega}|\varphi| + C\max_{\Omega}|f| \quad \text{for any } x \in \Omega$$

where C is a positive constant depending only on λ, Λ and diam (Ω).

Proof

We will construct a function w in Ω such that

(i) $L(w\pm u)=Lw\pm f\leq 0$, or $Lw\leq\mp f$ in Ω;

(ii) $w \pm u = w \pm \varphi \geq 0$, or $w \geq \mp\varphi$ on $\partial\Omega$

Denote $F = \max_{\Omega}|f|$ and $\phi = \max_{\partial\Omega}|\varphi|$. We need

$$Lw\leq -F \text{ in } \Omega$$
$$w \geq \Phi \text{ on } \partial\Omega$$

Suppose the domain Ω lies in the set $\{0 < x_1 < d\}$ for some $d > 0$. Set $w = \Phi + (e^{\alpha d} - e^{\alpha x_1})F$ with $\alpha > 0$ to be chosen later. Then by direct calculation, we have,

$$-Lw = (a_{11}\propto^2 + b_1\propto)Fe^{\alpha x_1} - c^{\Phi} - c(e^{\alpha d} - e^{\alpha x_1})F$$
$$\geq (a_{11}\propto^2 + b_1\propto)F \geq (\propto^2\lambda + b_1\propto)F \geq F$$

By choosing $\propto$ large such that $\propto^2\lambda + b_1(x)\propto \geq 1$ for any $x \in \Omega$. Hence w satisfies (i) and (ii). By the comparism principle, we conclude $-w \leq u \leq w$ in Ω, that is,

$$\sup_{\Omega}|u| \leq \Phi + (e^{\alpha d} - 1)F$$

where $\alpha$is a positive constant depending only on λ and Λ

(2) Gradient Estimates

The basic idea in the treatment of gradient estimates, accrued to Bernstein involves differentiation of the equation with respect to x_k, $k = 1, ..., n$, followed by multiplication by D_ku and summation over k. The maximum principle is then applied to the resulting equation in the function $v = |Du|^2$ possibly with some modification. The two kinds of gradient estimates are global gradient estimates and interior gradient estimates. We will use semilinear equations to illustrate the idea.

Suppose Ω is a bounded and connected domain in R^n. Consider the equation

$a_{ij}(x)D_{ij}u +$

$b_i(x)D_iu = f(x, u)$ in Ω (7)

for $u \in C^2(\Omega)$ and $f \in C(\Omega \times R)$. a_{ij} and b_i are always assumed to be continuous and hence bounded in $\bar{\Omega}$ and the equation is uniformly elliptic in Ω in the sense

$a_{ij}(x)\xi_i\xi_j \geq \lambda\, |\xi|^2$ for any $x \in \Omega$ and any $\xi \in R^n$ for some positive constant λ.

Proposition 9: Suppose $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies (7) for $a_{ij}, b_i \in C^1(\bar{\Omega})$ and $f \in C^1(\bar{\Omega} \times R)$. Then there holds

$\sup_{\Omega}|u| \leq \sup_{\partial\Omega}|u| + C$

Where C is a positive constant depending only on λ, diam(Ω) , $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L_\infty(\Omega)}$, and $|f|_{C^1(\bar{\Omega} \times [-M,M])}$

Proof

Set $L \equiv a_{ij} D_{ij} + b_i D_i$. We calculate $L(|Du|^2)$ first.

Note

$$D_i(|Du|^2) = 2D_k u D_{ki} u$$

and

$$D_{ij}(|Du|^2) = 2D_{ki}uD_{kj}u + 2D_kuD_{kij}u$$

(8)

Differentiating (8) with respect to x_k, multiplying by $D_k u$, and summing over k, we have by (8)

$$a_{ij}D_{ij}(|Du|^2) + b_i D_i(|Du|^2) = 2 a_{ij}D_{ki}uD_{kj}u - 2D_k a_{ij} D_k u D_{ij} u - 2D_k b_i$$

$$D_k u D_i u + 2D_z f|Du|^2 + 2D_k f D_k u$$

The ellipticity assumption means

$$\sum_{i,j,k} a_{ij} D_{ki}uD_{kj}u \geq \lambda\left|D^2u\right|^2.$$

By Cauchy inequality.

$$L(|Du|^2) \geq \lambda\left|D^2u\right|^2 - C\left|D^2u\right|^2 - C$$

Where C is a positive constant depending only on λ, $\left|a_{ij}, b_i\right|_{C^1(\bar{\Omega})}$, and $|f|_{C^1(\bar{\Omega} \times [-M,M])}$.

We need to add another term u^2. We have by ellipticity assumption

$$L(u^2) = 2a_{ij}D_iuD_ju + 2u\left\{a_{ij}D_{ij}u + b_iD_iu\right\} \geq 2\lambda|Du|^2 + 2uf$$

We then obtain

$$L(|Du|^2 + \alpha u^2) \geq \lambda\left|D^2u\right|^2 + (2\lambda\alpha - C)|Du|^2 - C$$

$$\geq \lambda\left|D^2u\right|^2 + |Du|^2 - C$$

if we choose $\propto > 0$ large, with C depending in addition on M. In order to control the constant term, we may consider another function $e^{\beta x_1}$ for $\beta > 0$. Hence we get

$$L(|Du|^2 + \alpha u^2 + e^{\beta x_1}) \geq \lambda\left|D^2u\right|^2 + |Du|^2 + \left\{\beta^2 a_{11} e^{\beta x_1} + \beta b_1 e^{\beta x_1} - C\right\}.$$

If we putthe domain $\Omega \subset \{x_1 > 0\}$, then $e^{\beta x_1} \geq 1$ for any $x \in \Omega$. By choosing βlarge, we may make the last term positive.Therefore, if we set $w = |Du|^2 + \alpha|u|^2 + e^{\beta x_1}$ for large α, β depending only on λ, diam(Ω), $\left|a_{ij}, b_i\right|_{C^1(\bar{\Omega})}$, $M = |u|_{L_\infty(\Omega)}$, and $|f|_{C^1(\bar{\Omega} \times [-M,M])}$, then we obtain $Lw \geq 0$ in Ω

By the maximum principle, we have

$\sup_\Omega w \leq \sup_{\partial\Omega} w$

V. CONCLUSION

The maximum principle are powerful tools in the analysis of elliptic partial differential equations, allowing for conclusions about existence, uniqueness,and properties of solutions. Their applications extend to various fields, including mathematics, physics, engineering,and geometry, where understanding the behavior of solutions is crucial. In mathematics, the maximum principle is applicable to a priori estimates and gradient estimates.

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