On Some Properties of Multigroups

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ABSTRACT- In this paper, we studied some algebraic properties of multigroups. Some of the properties of multisets have also been studied. We discovered that for any multiset G to be a multigroup over the group X, then its count function C_G must satisfies the two conditions: (i) $C_G(xy) \geq [C_G(x) \wedge C_G(y)], \forall\, x, y \in X$; (where $\wedge$ is the minimum operation). (ii) $C_G(x^{-1}) \geq C_G(x)$, $\forall\, x \in X$. We also discovered that if X is a group and $A, B \in MG(X)$, then $A \cap B \in MG(X)$ but $A \cup B \notin MG(X)$. It has also been shown that if A and B are two multigroups over a group X, then A is said to be a submultigroup of B if $\mathrm{A} \subseteq \mathrm{B}$. Hence, the study shows that the theory of multisets and multigroups can be very useful in many areas such as information retrieval on the web, data mining, decision making, data encryption, coding theory etc.

Keywords - Multisets, Group, Multigroups, Count function.

I. INTRODUCTION

In mathematics, the concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right.

Various physical systems, such as crystals and the hydrogen atom, may be modelled by symmetry groups. Thus group theory and the closely related representation theory have many important applications in physics, chemistry, and materials science. Group theory is also central to public key cryptography [3].

A multigroup $\mathfrak{R}$ is an algebraic system with one operation called multiplication. This multiplication usually satisfies the ordinary group axioms except that the product is not unique. Various problems in non-commutative algebra lead naturally to the introduction of algebraic systems in which the operations are not one-valued [6].

II. LITERATURE REVIEW

According to Shinoj and Sunil [7], Abstract algebra is the study of algebraic structures and more specifically the term algebraic structure generally refers to a set (called carrier set or underlying set) with one or more operations defined on it. Examples of algebraic structures include groups, rings, fields, and lattices. They introduced algebraic structures on Fuzzy multisets by extending these algebraic structures on Intuitionistic Fuzzy multisets to a new concept named Intuitionistic Fuzzy multigroups.

[4], introduced the notion of multigroups and also studied some basic results regarding multisets, like functional image and pre-image of a multiset under a mapping, decomposition theorems of multisets etc.

Various problems in non-commutative algebra lead naturally to the introduction of algebraic systems in which the operations are one-valued. [6], refers to multigroup $\mathfrak{R}$ as an algebraic system with one operation called multiplication. This multiplication usually satisfies the ordinary group axioms except that the product is not unique.

Another contribution to this theory was made by [8], where he referred to a hypergroup as a system in which any two elements a, b can be combined to form the product ab, which is a complex of n not necessarily distinct elements of the system. Here, n is a fixed integer ≥ 1. If $n = 1$, the hypergroup reduces to an ordinary group. If $[ab]$, called the bracket product, is the set of all the distinct elements of ab, the totality of elements a such that $[ax]$ and $[xa]$ are single elements for every x forms a group with respect to the bracket product. This group is called the nucleus.

The history of the theory of multigroups is very short. Multigroups (or hypergroups) were first defined by [5], who has studied their properties and applications in several communications.

It has been shown that the intersection of two multigroups is again a multigroup but their union may not be a multigroup. In this paper, we shall review some basic results regarding multigroup as it relates to multiset and define notions such as normal multigroup, factor multigroup, abelian multigroup etc., and again study some of their basic properties.

III. AIM AND OBJECTIVES

The aim of this research is to study the notion of multigroups as it relates to multisets. The specific objectives include, to:

i. investigate the conditions to which a multiset over a given set say X becomes a multigroup over X.
ii. prove some results/ propositions in multigroup theory.
iii. solve examples of multigroup problems.

IV. METHODOLOGY

In this section, we shall be looking at some definitions and propositions on multigroups as used in [1] and [2], as well as to prove some results on multigroups.

4.1 Results on Intersection of Multigroups

Definition 4.1.1. Let X be a group. A multiset G over X is said to be a multigroup over X if the count function of G (written as C_G) satisfies the following two conditions:
(i) $C_G(xy) \geq [C_G(x) \wedge C_G(y)], \forall x, y \in X$; (where $\wedge$ is the minimum operation).
(ii) $C_G(x^{-1}) \geq C_G(x), \forall x \in X$.

The set of all multigroups over X is denoted by MG(X).
Definition 4.1.2. Let $A, B \in [X]^w$. Then we define $A \circ B$ and A^{-1} as follows: $C_{A\circ B}(x) = \vee \{C_A(y) \wedge C_B(z);\ y, z \in X \text{ and } yz = x\}$ and $C_{A^{-1}}(x) = C_A(x^{-1})$. Where, $\vee$ is maximum operation and $\wedge$ is minimum operation.
Definition 4.1.3. Let X and Y be two nonempty sets and $f: X \to Y$ be a mapping. Then
(i) the image of a multiset $M \in [X]^w$ under the mapping f is denoted by f(M) or f[M], where $C_{f(M)}(y) = \begin{cases} \vee_{f(x)=y} C_M(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$
(ii) the inverse image of a multiset $N \in [Y]^w$ under the mapping f is denoted by
$f^{-1}(N)$ or $f^{-1}[N]$ where $C_{f^{-1}(N)}(X) = C_N[f(x)]$.

Proposition 4.1.4. Let $A \in MG(X)$. Then

(i) $C_A(e) \geq C_A(x), \forall x \in X$; (e is an identity element)
(ii) $C_A(x^n) \geq C_A(x), \forall x \in X$;

(iii) $C_A(x^{-1}) = C_A(x), \forall x \in X$;

(iv) $A = A^{-1}$.
Proof.

Let $x, y \in G$.

(i) $C_A(e) = C_A(xx^{-1}) \geq [C_A(x) \wedge C_A(x^{-1})]$.

By definition of multigroups, we have that

$C_A(e) \geq [C_A(x) \wedge C_A(x)] = C_A(x), \forall x \in X$;

Hence, $C_A(e) = C_A(x)$.

(ii)$C_A(x^n) = C_A(xx^{n-1}) \geq [C_A(x) \wedge C_A(x^{n-1})] = [C_A(x) \wedge C_A(xx^{n-2})] \geq [C_A(x) \wedge C_A(x) \wedge C_A(x^{n-2})] \geq [C_A(x) \wedge C_A(x) \wedge ... \wedge C_A(e)]$.

But $C_A(e) = C_A(x)$.

Therefore $C_A(x^n) \geq [C_A(x) \wedge C_A(x) \wedge ... \wedge C_A(x)] = C_A(x), \forall x \in X$
Hence, $C_A(x^n) \geq C_A(x)$.
(iii) Since $C_A(x^{-1}) \geq C_A(x) = C_A([x^{-1}]^{-1})$.

By definition of multigroups, $C_A([x^{-1}]^{-1}) \geq C_A(x^{-1}) \geq C_A(x)$.
Hence, $C_A(x^{-1}) = C_A(x), \forall x \in X$.

(iv) Since $C_{A^{-1}}(x) = C_A(x^{-1}) = C_A(x)$, comparing subscript (i.e. for $C_{A^{-1}}(x) = C_A(x)$), we have that

$$A = A^{-1}.$$

Hence, $A = A^{-1}$. End of proof.

Proposition 4.1.5. Let A be a multiset. Then $A \in MG(X)$ if and only if $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)], \forall x, y \in X$.

Proof.

Let $A \in MG(X)$. Then $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y^{-1})]$
but $C_A(y^{-1}) = C_A(y), \ \forall\, y \in X$,

$\Rightarrow \ C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)], \ \forall\, x, y \in X.$

Therefore, the given condition is satisfied.

Conversely let the given condition be satisfied. Now,

$C_A(e) = C_A(xx^{-1}) \geq [C_A(x) \wedge C_A(x)] = C_A(x), \ \forall\, x, \in X.$
4.1 Applying equation 4.1, we have
Again $C_A(x^{-1}) = C_A(ex^{-1}) \geq [C_A(e) \wedge C_A(x)] = C_A(x), \forall\, x, \in X$
4.2 Also $C_A(xy) = C_A[x(y^{-1})^{-1}] \geq [C_A(x) \wedge C_A(y^{-1})]$
but $C_A(y^{-1}) = C_A(y), \ \forall\, y \in X$,
Applying equation 4.2, we have

$\Rightarrow \ C_A(xy) \geq [C_A(x) \wedge C_A(y)], \ \forall\, x, y \in X$
4.3 Therefore, from equation 4.2 and 4.3, we have that $A \in MG(X)$.
Proposition 4.1.6. If $A \in MS(X)$. Show that

$A \in MG(X)$ if and only if $A \circ A^{-1} = A$.
Proof.
Let $C_A(x) \geq C_A(y) \wedge C_A(z), \ \forall\, y, z \in X$

$C_A(x) \geq \vee_{y,z \in X} \{C_A(y) \wedge C_A(z) : yz = x; \ \forall\, x \in X\}$

$C_A(x) \geq C_A(y) \wedge C_{A^{-1}}(z^{-1})$

Recall that $C_G(x^{-1}) \geq C_G(x)$

$\Rightarrow \ C_A(x) \geq C_A(y) \wedge C_{A^{-1}}(z)$

$C_A(x) \geq [C_A(y) \wedge C_{A^{-1}}(z)] = C_{A \circ A^{-1}}(x)$ since $yz = x$
$\Rightarrow \ C_A(x) \geq C_{A \circ A^{-1}}(x)$

By comparing subscript, we have that,

$A \geq A \circ A^{-1}$
(*) Conversely,
$C_{A \circ A^{-1}}(yz) \geq C_A(y) \wedge C_{A^{-1}}(z); \ \forall\, y, z \in X$

$C_{A \circ A^{-1}}(x) \geq \vee_{y,z \in X} \{C_A(y) \wedge C_{A^{-1}}(z) : yz = x; \ \forall\, x \in X\}$
$\Rightarrow \ C_{A \circ A^{-1}}(x) \geq C_A(y) \wedge C_A(z^{-1})$

Recall that $C_G(x^{-1}) \geq C_G(x)$, therefore we have that

$C_{A \circ A^{-1}}(x) \geq C_A(y) \wedge C_A(z) : yz = x; \ \forall\, x \in X$

$\Rightarrow \ C_{A \circ A^{-1}}(x) \geq C_A(x)$ since $yz = x$

By comparing subscript, we have that,

$A \circ A^{-1} \geq A$
(**) (*) and (**) implies that

$A \circ A^{-1} = A$
Hence, $A \in MG(X)$. End of proof.
Proposition 4.1.7. Let $A \in MS(X)$. Then $A \in MG(X)$ if and only if A satisfies the following conditions:
(a) (i) $A \circ A \subseteq A$;

(ii) $A^{-1} \subseteq A$ or $A \subseteq A^{-1}$ or $A^{-1} = A$.

Or

(b) $A \circ A^{-1} \subseteq A$.

Proof.

Let $A \in MG(X)$. Then $C_A(yz) \geq [C_A(y) \wedge C_A(z), \ \forall\, y, z \in X$. Thus, $C_A(x) \geq \{C_A(y) \wedge +C_A(z) : yz = x\}$. Hence

$C_A(x) \geq \vee_{y,z \in X} \{C_A(y) \wedge C_A(z) : yz = x\} = C_{A \circ A}(x), \ \forall\, x \in X.$
Therefore, $A \circ A \subseteq A$. Again since $C_{A^{-1}}(x) = C_A(x^{-1}) = C_A(x)$, it follows that $A = A^{-1}$ and hence $A \subseteq A^{-1}$ and $A^{-1} \subseteq A$. Thus the given conditions are satisfied.
Conversely, let the given conditions be satisfied. Let $x, y \in X$. Then
$C_A(xy^{-1}) \geq C_{A \circ A}(xy^{-1}) = \vee_{z \in X} [C_A(z) \wedge C_A(z^{-1}xy^{-1})]$.
$\geq [C_A(x) \wedge C_A(y^{-1})] = [C_A(x) \wedge C_A(y)]$
Therefore, $A \in MG(X)$. End of proof.
Proposition 4.1.8. Let $A, B \in MG(X)$.

Then $A \circ B \in MG(X)$ if and only if $A \circ B = B \circ A$.

Proof.

Since $A, B \in MG(X)$, it follows that $A = A^{-1}$ and $B = B^{-1}$.

Suppose $A \circ B \in MG(X)$. Then $A \circ B = (A \circ B)^{-1} = B^{-1} \circ A^{-1} = B \circ A$.
Conversely, let $A \circ B = B \circ A$. Then $(A \circ B)^{-1} = (B \circ A)^{-1} = A^{-1} \circ B^{-1} = A \circ B$ and $(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B = A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B) \subseteq A \circ B$. Therefore, $A \circ B \in MG(X)$.
End of proof.
Proposition 4.1.9. Let $A, B \in MG(X)$. Then $A \cap B \in MG(X)$.
Proof.

Since $A, B \in MG(X)$, we have $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)]$ and $C_B(xy^{-1}) \geq [C_B(x) \wedge C_B(y)]$, $\forall\, x, y \in X$. Now

$$C_{A\cap B}(xy^{-1}) = \wedge\{C_A(xy^{-1}),\ C_B(xy^{-1})\}$$

$$\geq \wedge\{[C_A(x) \wedge C_A(y^{-1})],\ [C_B(x) \wedge C_B(y^{-1})]\}$$

By definition of multigroups, we have that

$$C_{A\cap B}(xy^{-1}) \geq \wedge\{[C_A(x) \wedge C_A(y)],\ [C_B(x) \wedge C_B(y)]\}$$
$$= C_A(x) \wedge C_A(y) \wedge C_B(x) \wedge C_B(y)$$

$$= [C_A(x) \wedge C_B(x)] \wedge [C_A(y) \wedge C_B(y)]$$

$$= C_{A\cap B}(x) \wedge C_{A\cap B}(y)$$

and $C_{A\cap B}(x^{-1}) = C_A(x^{-1}) \wedge C_B(x^{-1})$

$$= C_A(x) \wedge C_B(x) = C_{A\cap B}(x).$$

Therefore, $A \cap B \in MG(X)$. End of proof.

4.2. Results on Subgroups of Multigroups
Proposition 4.2.1. Let $A \in MG(X)$. Then A_n, $n \in \mathbb{N}$ are subgroups of X.
Proof.

Let $x, y \in A_n$. Then $C_A(x) \geq n$ and $C_A(y) \geq n$.
Since $A \in MG(X)$, it follows that $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y^{-1})]$

By definition of multigroups, we have

$C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)] \geq n$. Hence, $xy^{-1} \in A_n$.
Therefore, A_n, $n \in \mathbb{N}$ are subgroups of X. End of proof.

Proposition 4.2.2. If $A \in MG(X)$ and $H \leq X$, then $A/_H \in MG(H)$.
Proof.
Let $x, y \in H$. Then $xy^{-1} \in H$. Since $A \in MG(X)$, then
$C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y^{-1})$.
By definition of multigroups, we have that
$C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y)$; $\forall\, x, y \in X$.
Moreover, $C_{A/_H}(xy^{-1}) \geq C_{A/_H}(x) \wedge C_{A/_H}(y^{-1})$.
By definition of multigroups, we have that
$C_{A/_H}(xy^{-1}) \geq C_{A/_H}(x) \wedge C_{A/_H}(y)$; $\forall\, x, y \in X$
Hence, $A/_H \in MG(H)$.
Definition 4.2.3. Let $A \in MG(X)$. Then define $A^* = \{x \in X;\ C_A(x) = C_A(e)\}$ and $A_* = \{x \in X;\ C_A(x) > 0\}$.
Proposition 4.2.4. Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.

Proof.

Let $x, y \in A^*$. Then $C_A(x) = C_A(y) = C_A(e)$. Now $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)] = [C_A(e) \wedge C_A(e)] = C_A(e) \geq C_A(xy^{-1})$.
So, $C_A(xy^{-1}) = C_A(e)$, $\forall\, x, y \in X$ and hence $x, y \in A^* \Longrightarrow xy^{-1} \in A^*$. Therefore, A^* is a subgroup of X.

Again let $x, y \in A_*$. Then $C_A(x) > 0$ and $C_A(y) > 0$. Now
$C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)] > 0$.

Therefore, $x, y \in A_* \Longrightarrow xy^{-1} \in A_*$ and hence A_* is a subgroup of X. End of proof.

V. RESULTS

Example 5.1. Let $X = \{e, a, b, c\}$ be Klein's 4-group and $G = \{e, e, e, a, a, b, b, b, c, c\}$ be a multiset over X. Show that G is also a multigroup over X.
Solution.
G will only be a multigroup over X if the count function G (C_G) satisfies the following two conditions:

(i) $C_G(xy) \geq [C_G(x) \wedge C_G(y)]$, $\forall\, x, y \in X$;

(ii) $C_G(x^{-1}) \geq C_C(x)$, $\forall\, x \in X$.
Now, $X = \{e, a, b, c\}$ gives
$\{e, a\}, \{e, b\}, \{e, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{e, e\}, \{a, a\}, \{b, b\}, \{c, c\}, \{e^{-1}\}, \{a^{-1}\}, \{b^{-1}\}, \{c^{-1}\}$
(i) $\Longrightarrow$ $C_G(ea) = C_G(a) = 2 \geq [C_G(e) \wedge C_G(a)]$,

$C_G(eb) = C_G(b) = 3 \geq [C_G(e) \wedge C_G(b)]$,

$C_G(ec) = C_G(c) = 2 \geq [C_G(e) \wedge C_G(c)]$,

$C_G(ab) = C_G(c) = 2 \geq [C_G(a) \wedge C_G(b)]$,

$C_G(ac) = C_G(b) = 3 \geq [C_G(a) \wedge C_G(c)]$,

$C_G(bc) = C_G(a) = 2 \geq [C_G(b) \wedge C_G(c)]$,

$C_G(ee) = C_G(e) = 3 \geq [C_G(e) \wedge C_G(e)]$,

$C_G(aa) = C_G(e) = 3 \geq [C_G(a) \wedge C_G(a)]$,

$C_G(bb) = C_G(e) = 3 \geq [C_G(b) \wedge C_G(b)]$,

$C_G(cc) = C_G(e) = 3 \geq [C_G(c) \wedge C_G(c)]$,

(ii) $\Longrightarrow \quad C_G(e^{-1}) = C_G(e) = 3$,

$C_G(a^{-1}) = C_G(a) = 2$,

$C_G(b^{-1}) = C_G(b) = 3$,

$C_G(c^{-1}) = C_G(c) = 2$.

Since condition (i) and (ii) are satisfied as clearly shown above, we therefore said that G is a multigroup over X.
Definition 5.2. Let A and B be two multigroups over a group X. Then A is said to be a submultigroup of B if $A \subseteq B$.
Example 5.3. Let $X = \{e, a, b, c\}$ be Klein's 4-group, $A = \{e, e, a, a, b, b, c, c\}$ and
$B = \{e, e, e, a, a, b, b, b, c, c\}$. Then clearly $A, B \in MG(X)$ and $A \subseteq B$. Therefore, A is a submultigroup of B.
Example 5.4. Let $X = K_4 = \{e, a, b, c\}$ be the Klein's 4-group, $A = \{e, e, a\}$ and $B = \{e, e, b\}$. If $A, B \in MG(X)$, show whether or not $A \cup B$, $A \cap B$ is a multigroup over X.
Solution.

$$\text{Let } G = A \cup B = \{e, e, a, b\}$$

G will only be a multigroup over X if the count function G (C_G) satisfies the following two conditions:
(i) $C_G(xy) \geq [C_G(x) \wedge C_G(y)], \quad \forall\, x, y \in X$;

(ii) $C_G(x^{-1}) \geq C_C(x), \quad \forall\, x \in X$.

Now, $G = \{e, e, a, b\}$ gives
$\{e, a\}, \{e, b\}, \{a, b\}, \{e, e\}, \{a, a\}, \{b, b\}, \{e^{-1}\}, \{a^{-1}\}, \{b^{-1}\}$

(i) $\Longrightarrow \quad C_G(ea) = C_G(a) = 1 \geq \wedge [C_G(e), C_G(a)]$,

$C_G(eb) = C_G(b) = 1 \geq \wedge [C_G(e), C_G(b)]$,

$C_G(ab) = C_G(c) = 0 \ngeq \wedge [C_G(a), C_G(b)]$,

$C_G(ee) = C_G(e) = 2 \geq \wedge [C_G(e), C_G(e)]$,

$C_G(aa) = C_G(e) = 2 \geq \wedge [C_G(a), C_G(a)]$,

$C_G(bb) = C_G(e) = 2 \geq \wedge [C_G(b), C_G(b)]$,

(ii) $\Longrightarrow \quad C_G(e^{-1}) = C_G(e) = 2$,

$C_G(a^{-1}) = C_G(a) = 1$,

$C_G(b^{-1}) = C_G(b) = 1$,

Since condition (i) is not satisfied as clearly shown above, i.e. $A \cup B = \{e, e, a, b\}$ and $C_{A\cup B}(c) = C_{A\cup B}(ab) = 0 \ngeq \wedge [C_{A\cup B}(a), C_{A\cup B}(b)] = 1$.

We therefore said that $A \cup B \notin MG(X)$.
But $A \cap B = \{e, e\}$ and $C_{A\cap B}(e) = 2 \geq \wedge [C_{A\cap B}(e), C_{A\cap B}(e)]$. Therefore, $A \cap B$ but not $A \cup B$ is a multigroup over X.

VI. DISCUSSION

From example 4.1, we see that the Klein's 4-group X and multiset G satisfied the condition for multigroups, hence G is a multigroup over X. We also see from example 4.3 that the multigroup A is a submultigroup of the multigroup B since all the elements in the multiset A are all contained in the multiset B. Therefore, $A \subseteq B$ and A is a submultigroup of B. We verified in example 4.4 whether or not the union or intersection of the two multigroups A and B is a multigroup, but the result shows that only their intersection is a multigroup but their union is not a multigroup since it does not satisfied the conditions for multigroups.

VII. CONCLUSION

From the results obtained above, we see clearly that the union of family of multigroups over a group X may not be a multigroup, but the intersection of family of multigroups over a group X is a multigroup. Also, if A is a multiset over X, then A is a multigroup over the group X if and only if $A \circ A^{-1} = A$, or $A \circ A^{-1} \subseteq A$. Hence the study of multigroups is very interesting and useful in areas such as information retrieval on web (since an information may appear more than once with possibly different degrees of relevance to a query), coding theory, etc.

REFERENCES

[1] Arfur, D. (1998). An introduction to group theory. *http://www.marlboro.edu/~mahoney/groups/dog_school/groups.html*.

[2] Awolola, J. A. and Ibrahim, A. M. (2016). Some results on multigroups. *Quasigroups and related systems*. 24:169-177.

[3] Erno, R. (1974). Group theory. *Wikipedia*.

[4] Nazmul, S. K., Majumdar, P. and Samanta, S. K. (2013). On multisets and multigroups. *Annals of fuzzy mathematics and informatics*. 6(3): 643-656.

[5] Marty, F. (1934). Sur une generalization de la notion de groupe. *Huitieme congres des mathematiciens scandinaves, Stockholm*. 201:45-49.

[6] Melvin, D. and Oystein, O. (1938). Theory of multigroups. *American journal of mathematics*. 60(3): 705-733.

[7] Shinoj, T. K. and Sunil, J. J. (2015). Intuitionistic fuzzy multigroups. *Annals of pure and applied mathematics*. 9: 131-143.

[8] Wall, H. S. (1937). Hypergroups. *American journal of mathematics*. 59:77-98.