

Application of Adomian Decomposition Method for Solving Volterra Integral Equations

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Abstract- In this paper, firstly, definitions and classifications of integral equations are described. Next, Adomian decomposition method is observed. Moreover, Adomian decomposition method is set out with the aim of solving linear and nonlinear Volterra integral equations. Finally, some applications are expressed to demonstrate the effectiveness of Adomian decomposition methods for solving linear and nonlinear Volterra integral equations.

Keywords: Adomian Decomposition Method, Exact Solution, MATLAB, Series Solution, Volterra Integral Equation

I. INTRODUCTION

Integral equations play an important role in mathematical modeling by providing a powerful framework for describing relationship between functions in terms of integral. The problems in the piratical fields can be constructed by Mathematical models and Adomian decomposition method is used in order to get the solutions. These equations are applied in various filed such as physics, biology, economic and population growth. The unknown function $u(x)$ appears inside an integral sign is called an integral equation. The common type of integral equation in $u(x)$ is

$$u(x) = f(x) + \lambda \int_{p(x)}^{q(x)} \kappa(x, t) u(t) dt,$$

where the limits of integration are $p(x)$ and $q(x)$, $\kappa(x, t)$ which is called the kernel of the integral equation is a function of two variables x and t and λ is a constant parameter.

1.1 Definition

In Volterra integral equations, at least one of the limits of integration is a variable.

For the first kind Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x \kappa(x, t) u(t) dt. \quad (1)$$

However, in Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign in the form:

$$u(x) = f(x) + \lambda \int_0^x \kappa(x, t) u(t) dt. \quad (2)$$

The kernel $\kappa(x, t)$ and the function $f(x)$ are the given real-valued functions, and λ is a parameter. The second kind of Volterra integral equation can be classified homogeneous and inhomogeneous according to the value of the function $f(x)$. If $f(x)$ is identically zero, the equation is called homogeneous whereas it is called inhomogeneous or nonhomogeneous if the function $f(x) \neq 0$. The Volterra integral equations appear in operator theory and it can be derived from initial value problems with given initial conditions.

1.2 Linearity concept

If the exponent of the unknown function $u(x)$ inside the integral sign is one, it is called the linear integral equation. The integral equation is called nonlinear if the equation contains nonlinear functions of $u(x)$, such as e^u , $\cos u$, $\sinh u$, $\ln(1+u)$, or if the unknown function $u(x)$ has exponent other than one, the integral equation is defined nonlinear. The following examples are linear and nonlinear Volterra integral equations respectively.

$$u(x) = 1 - \int_0^x (x-t) u(t) dt, \text{ and}$$

$$u(x) = 1 + \int_0^x (1+x-t) u^4(t) dt.$$

II. ALGORITHM REPRESENTATION OF ADOMIAN DECOMPOSITION METHOD

In the 1980s, George Adomian found out Adomian decomposition method (ADM) which comprises decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (3)$$

or equivalently,

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \dots, \quad (4)$$

where the components $u_n(x), n \geq 0$ will be decided recursively. It deals with finding each component $u_0, u_1, u_2, \dots$. The resolution of each component can be evaluated through a recurrence relation which contains simple integral. We will tackle the Volterra integral equation and demonstrate how it can be handled by using ADM. The Volterra integral equation (2) is

$$u(x) = f(x) + \lambda \int_0^x \kappa(x, t) u(t) dt,$$

where a constant parameter is λ , the function $f(x)$ and the kernel $\kappa(x, t)$ are real-valued functions. The zero is the lower limit and the variable x is the upper limit. The series solution for the unknown function $u(x)$ of the Volterra integral equation (2) is obtained by using in series assumption (3). By the decomposition method, (3) is substituted into the Volterra integral equation (2) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x \kappa(x, t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt, \quad (5)$$

or equivalently,

$$u_0(x) + u_1(x) + u_2(x) + \dots = f(x) + \lambda \int_0^x \kappa(x, t) [u_0(t) + u_1(t) + \dots] dt. \quad (6)$$

All terms that are not contained under the integral sign are assigned the zeroth component $u_0(x)$. The rest components $u_j(x)$ for $j \geq 1$ are calculated by using a recurrence relation:

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = \lambda \int_0^x \kappa(x, t) u_n(t) dt, \quad n \geq 0,$$

or equivalently,

$$\left. \begin{aligned} u_0(x) &= f(x), \\ u_1(x) &= \lambda \int_0^x \kappa(x, t) u_0(t) dt, \\ u_2(x) &= \lambda \int_0^x \kappa(x, t) u_1(t) dt, \\ u_3(x) &= \lambda \int_0^x \kappa(x, t) u_2(t) dt, \\ u_4(x) &= \lambda \int_0^x \kappa(x, t) u_3(t) dt, \\ &\vdots \end{aligned} \right\} \quad (7)$$

Hence, the solution of each component $u_0(x), u_1(x), u_2(x), u_3(x), \dots$ are simply find by applying (7).

2.1 Types of solution

A solution of Volterra integral equation of second kind arise in any of the following two types.

a) Exact solution

The closed form solution such as polynomial, exponential, trigonometric functions or the combination of two elementary functions, converges to the exact is called the exact solution of the integral equation. For example:

$$u(x) = x^2 + e^x, \quad u(x) = \sin x + e^{3x},$$

$$u(x) = 8 + \sin 5x + \cos 5x - 2e^{-x} \text{ and many others.}$$

b) Series solution

Sometimes exact solutions can't be obtained for concrete problems. In this case, we have solution in the series form that may converge to exact solution if such a solution exists. In other words, series may not give exact solution and in this case, we have approximate solution for numerical purpose. The more terms in the series give the higher accuracy in the solution.

2.2 Nonlinear Volterra integral equations of the second kind

The second kind of the nonlinear Volterra integral equation is defined by

$$u(x) = f(x) + \lambda \int_0^x \kappa(x,t) N(u(t)) dt, \quad (8)$$

where a nonlinear function of $u(x)$ is $N(u(t))$. The nonlinear Volterra integral equation (8) contains the linear term $u(x)$ and the nonlinear function $N(u(t))$. The linear term $u(x)$ of (8) can be represented normally by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

where the components $u_n(x), n \geq 0$ can be computed in a recursive manner. The nonlinear function $N(u(t))$ of (8) is presented by the Adomian polynomial series

$$N(u(t)) = u^n(t) = \sum_{n=0}^{\infty} A_n(t), \quad (9)$$

by using the algorithm

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^n \lambda^i u_i \right) \right]_{\lambda=0}, n = 0, 1, 2, \dots$$

Substituting (3) and (9) into (8) gives

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x \kappa(x,t) \left(\sum_{n=0}^{\infty} A_n(t) \right) dt.$$

The recurrence relation is used to determine the components

$$\left. \begin{aligned} u_0(x) &= f(x), \\ u_1(x) &= \lambda \int_0^x \kappa(x,t) A_0(t) dt, \\ u_2(x) &= \lambda \int_0^x \kappa(x,t) A_1(t) dt, \\ &\vdots \\ u_{n+1}(x) &= \lambda \int_0^x \kappa(x,t) A_n(t) dt, k \geq 0. \end{aligned} \right\} \quad (10)$$

In the view of (10), the solution of each component $u_0(x), u_1(x), u_2(x), \dots$ are absolutely find. As a result, the solution $u(x)$ of (8) in a series form is obtained by using the decomposition (3).

III. SOLVING LINEAR AND NONLINEAR VOLTERRA INTEGRAL EQUATIONS WITH ADOMIAN DECOMPOSITION METHOD

In this section, some applications are shown in order to display the effectiveness of Adomian decomposition method to obtain the solution of the second kind of linear and nonlinear Volterra integral equations.

3.1 Example

We consider the second kind of Volterra integral equation

$$u(x) = x + \frac{1}{2}x^2 + x^4 + \frac{1}{5}x^5 - \int_0^x u(t) dt. \quad (11)$$

It is clear that

$$f(x) = x + \frac{1}{2}x^2 + x^4 + \frac{1}{5}x^5, \quad \lambda = -1.$$

Substituting the decomposition series (3) into (11) gives

$$\sum_{n=0}^{\infty} u_n(x) = x + \frac{1}{2}x^2 + x^4 + \frac{1}{5}x^5 - \int_0^x \sum_{n=0}^{\infty} u_n(t) dt,$$

$$u_0(x) + u_1(x) + u_2(x) + \dots = x + \frac{1}{2}x^2 + x^4 + \frac{1}{5}x^5 - \int_0^x [u_0(t) + u_1(t) + u_2(t) + \dots] dt.$$

The zeroth component $u_0(x)$ is assumed by all terms which are not contained under the integral sign. Therefore, the following recurrence relations are obtained.

$$u_0(x) = x + \frac{1}{2}x^2 + x^4 + \frac{1}{5}x^5, \text{ and}$$

$$u_{n+1}(x) = -\int_0^x u_n(t) dt, n \geq 0,$$

that gives

$$u_0(x) = x + \frac{1}{2}x^2 + x^4 + \frac{1}{5}x^5,$$

$$u_1(x) = -\int_0^x u_0(t) dt = -\frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{5}x^5 - \frac{1}{30}x^6,$$

$$u_2(x) = -\int_0^x u_1(t) dt = \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^6 + \frac{1}{210}x^7,$$

$$u_3(x) = -\int_0^x u_2(t) dt = -\frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{210}x^7 - \frac{1}{1680}x^8,$$

and so on.

Using the Adomian decomposition method gives the series solutions

$$u(x) = \left(x + \frac{1}{2}x^2 + x^4 + \frac{1}{5}x^5 \right) + \left(-\frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{5}x^5 - \frac{1}{30}x^6 \right)$$

$$+\left(\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{30}x^6+\frac{1}{210}x^7\right)+\left(-\frac{1}{24}x^4-\frac{1}{120}x^5-\frac{1}{210}x^7-\frac{1}{1680}x^8\right)+\dots$$

By canceling the noise terms, the series solutions converge to the exact solution $u(x) = x + x^4$.

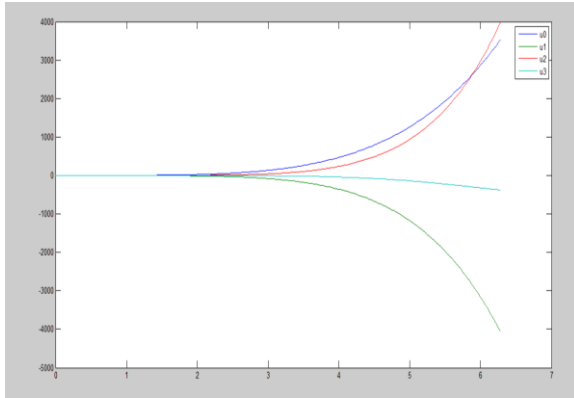


Figure 1. Graph of each component solutions

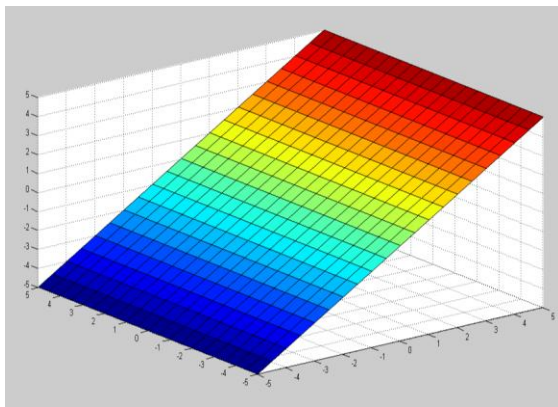


Figure 2. Graph of the exact solution

3.2 Example

The linear Volterra integral equation

$$u(x) = 1 + 2 \int_0^x t u(t) dt. \quad (12)$$

is considered.

It is known that $f(x) = 1$, $\lambda = 2$, and $\kappa(x, t) = t$.

Substituting the decomposition series (3) into (12) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 + 2 \int_0^x t \sum_{n=0}^{\infty} u_n(t) dt,$$

$$u_0(x) + u_1(x) + u_2(x) + \dots = 1 + 2 \int_0^x t [u_0(t) + u_1(t) + u_2(t) + \dots] dt.$$

Applying the recurrence relation, the components

$u_0(x) = 1$, and $u_{n+1}(x) = 2 \int_0^x t u_n(t) dt, k \geq 0$ are generally observed.

This carries out the solution of each component:

$$u_0(x) = 1,$$

$$u_1(x) = 2 \int_0^x t u_0(t) dt = x^2,$$

$$u_2(x) = 2 \int_0^x t u_1(t) dt = 2 \int_0^x t t^2 dt = \frac{1}{2} x^4,$$

$$u_3(x) = 2 \int_0^x t u_2(t) dt = \frac{x^6}{6},$$

$$u_4(x) = 2 \int_0^x t u_3(t) dt = \frac{x^8}{24},$$

$\vdots$

Hence, the series solution

$$u(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

is obtained.

This converges to the exact solution $u(x) = e^{x^2}$.

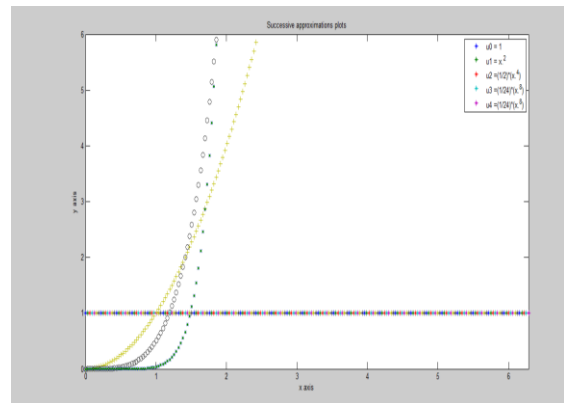


Figure 3. Graph of each component solutions

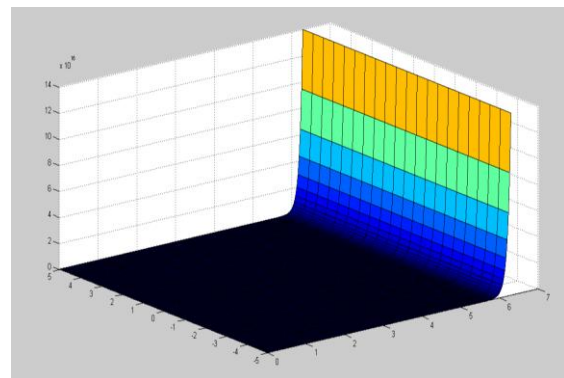


Figure 4. Graph of the exact solution

3.3 Example

The nonlinear Volterra integral equation

$$u(x) = e^x - x e^x + \int_0^x e^{x-2t} u^2(t) dt \quad (13)$$

is considered. (13)

Substituting the decomposition series (3) and (9) into the left side and right side of (13) respectively, it gives

$$\sum_{n=0}^{\infty} u_n(x) = e^x - xe^x + e^x \int_0^x e^{-2t} A_n(t) dt,$$

where A_n are the Adomian polynomials.

The first three Adomian polynomials for $u^2(x)$ are computed to be:

$$A_0 = N(u_0),$$

$$A_1 = u_1 N'(u_0),$$

$$A_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0),$$

$$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0),$$

and so on.

The Adomian decomposition method allows the recurrence relation

$$u_0(x) = e^x - xe^x,$$

$$u_{n+1}(x) = e^x \int_0^x e^{-2t} A_n(t) dt, \quad n \geq 0.$$

This gives

$$u_0(x) = e^x - xe^x,$$

$$A_0 = N(u_0) = u_0^2 = (e^x - xe^x)^2 \\ = e^{2x} - xe^{2x} - xe^{2x} + x^2 e^{2x}$$

$$u_1(x) = e^x (x - x^2 + \frac{1}{3} x^3),$$

$$A_1 = u_1 N'(u_0) = 2u_0 u_1$$

$$= 2(e^x - xe^x) e^x (x - x^2 + \frac{1}{3} x^3) \\ = 2e^{2x} (x - x^2 + \frac{1}{3} x^3 - x^2 + x^3 - \frac{1}{3} x^4) \\ = 2e^{2x} (x - 2x^2 + \frac{4}{3} x^3 - \frac{1}{3} x^4).$$

$$u_2(x) = e^x \int_0^x e^{-2t} A_1(t) dt$$

$$u_2(x) = e^x \int_0^x e^{-2t} 2e^{2t} (t - 2t^2 + \frac{4}{3} t^3 - \frac{1}{3} t^4) dt \\ = 2e^x \int_0^x (t - 2t^2 + \frac{4}{3} t^3 - \frac{1}{3} t^4) dt \\ = 2e^x \left[\frac{1}{2} t^2 - \frac{2}{3} t^3 + \frac{4}{3} \frac{t^4}{4} - \frac{1}{3} \frac{t^5}{5} \right]_0^x \\ = 2e^x \left(\frac{1}{2} x^2 - \frac{2}{3} x^3 + \frac{1}{3} x^4 - \frac{1}{15} x^5 \right)$$

$$u_2(x) = e^x \left(x^2 - \frac{4}{3} x^3 + \frac{2}{3} x^4 - \frac{2}{15} x^5 \right),$$

$$A_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0) = 2u_0 u_2 + u_1^2$$

$$= 2(e^x - xe^x) \left(e^x (x^2 - \frac{4}{3} x^3 + \frac{2}{3} x^4 - \frac{2}{15} x^5) \right) \\ + \left(e^x (x - x^2 + \frac{1}{3} x^3) \right)^2$$

$$= 2e^{2x} \left(x^2 - \frac{7}{3} x^3 + 2x^4 - \frac{12}{15} x^5 + \frac{2}{15} x^6 \right) + \\ e^{2x} \left(x^2 - 2x^3 + \frac{5}{3} x^4 - \frac{2}{3} x^5 + \frac{1}{9} x^6 \right) \\ = e^{2x} \left(3x^2 - \frac{20}{3} x^3 + \frac{17}{3} x^4 - \frac{34}{15} x^5 + \frac{17}{45} x^6 \right),$$

$$u_3(x) = e^x \int_0^x e^{-2t} A_2(t) dt$$

$$= e^x \int_0^x e^{-2t} e^{2t} (3t^2 - \frac{20}{3} t^3 + \frac{17}{3} t^4 - \frac{34}{15} t^5 + \frac{17}{45} t^6) dt \\ = e^x \left[t^3 - \frac{5}{3} t^4 + \frac{17}{15} t^5 - \frac{17}{45} t^6 + \frac{17}{315} t^7 \right]_0^x$$

$$u_3(x) = e^x (x^3 - \frac{5}{3} x^4 + \frac{17}{15} x^5 - \frac{17}{45} x^6 + \frac{17}{315} x^7),$$

$$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0) \\ = u_0 u_3 + 2u_1 u_2 + u_3 u_0$$

$$A_3 = e^{2x} (4x^3 - 10x^4 + \frac{154}{15} x^5 - \frac{248}{45} x^6 + \frac{496}{315} x^7 - \frac{62}{315} x^8)$$

$$u_4(x) = e^x \int_0^x e^{-2t} A_3(t) dt$$

$$= e^x \int_0^x (4t^3 - 10t^4 + \frac{154}{15} t^5 - \frac{248}{45} t^6 + \frac{496}{315} t^7 - \frac{62}{315} t^8) dt \\ = e^x \left[t^4 - 2t^5 + \frac{77}{45} t^6 - \frac{248}{315} t^7 + \frac{62}{315} t^8 - \frac{62}{2835} t^9 \right]_0^x$$

$$u_4(x) = e^x (x^4 - 2x^5 + \frac{77}{45} x^6 - \frac{248}{315} x^7 + \frac{62}{315} x^8 - \frac{62}{2835} x^9),$$

In the same way the other components can be found.

Hence, The series solutions

$$u(x) = e^x (1 - x^5 + \frac{4}{3} x^6 - \frac{11}{15} x^7 + \frac{62}{315} x^8 - \frac{62}{2835} x^9 + \dots)$$

is obtained. This converges to the exact solution

$$u(x) = e^x.$$

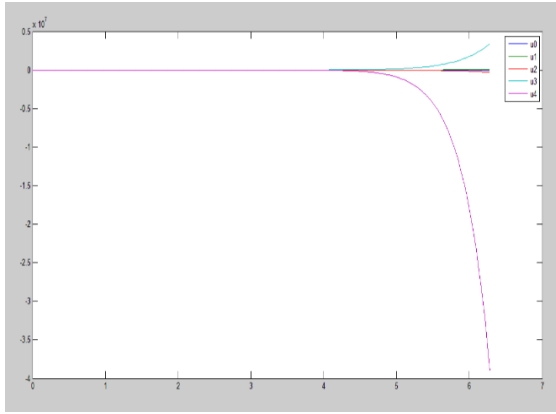


Figure 5. Graph of each component solutions

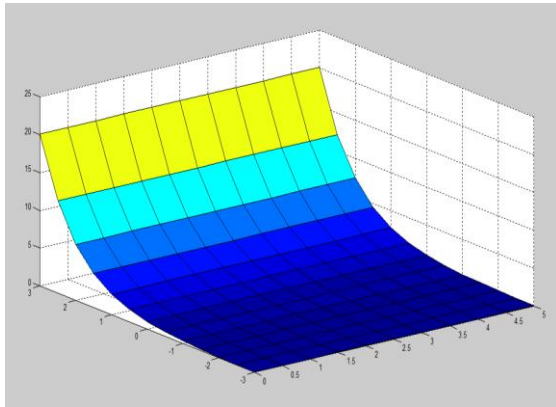


Figure 6. Graph of the exact solution

3.4 Example

The nonlinear Volterra integral equation

$$u(x) = x + \int_0^x u^2(t) dt \quad (14)$$

is considered.

Substituting the decomposition series (3) and (9) into (14), we find

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x A_n(t) dt,$$

where $A_n, n \geq 0$ represents the Adomian polynomials for nonlinear function, $u^2(x)$. The first three polynomials for $u^2(x)$ are computed to be:

$$A_0 = N(u_0) = u_0^2,$$

$$A_1 = u_1 N'(u_0) = 2u_0 u_1,$$

$$A_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0) = 2u_0 u_2 + u_1^2,$$

$$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0)$$

$$= u_0 u_3 + 2u_1 u_2 + u_3 u_0,$$

and so on.

The Adomian decomposition method allows the recurrence relation

$$u_0(x) = x, \text{ and } u_{n+1}(x) = \int_0^x A_n(t) dt, \quad n \geq 0.$$

This carries out the solution of each component:

$$u_0(x) = x,$$

$$u_1(x) = \int_0^x u_0^2(t) dt = \int_0^x t^2 dt = \frac{1}{3} x^3,$$

$$u_2(x) = \int_0^x (2u_0(t)u_1(t)) dt = \int_0^x \frac{2}{3} t^4 dt = \frac{2}{15} x^5,$$

$$u_3(x) = \int_0^x (2u_0(t)u_2(t) + u_1^2(t)) dt = \int_0^x \frac{51}{135} t^6 dt = \frac{17}{315} x^7,$$

$$u_4(x) = \int_0^x (2u_0(t)u_3(t) + 2u_1(t)u_2(t)) dt = \int_0^x \frac{62}{135} t^8 dt = \frac{62}{2835} x^9,$$

and so on.

Hence, the series solution

$$u(x) = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \dots,$$

that converges to the exact solution

$$u(x) = \tan x \text{ is obtained.}$$

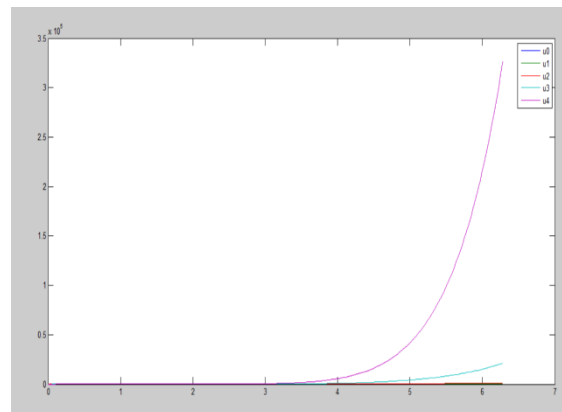


Figure 7. Graph of each component solutions

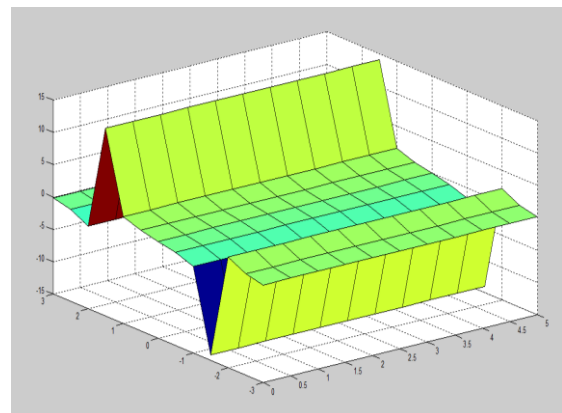


Figure 8. Graph of the exact solution

CONCLUSION

In this research paper, Adomian decomposition method is applied widely in Applied Mathematics and in the field of series solutions. This method is a useful method for solving linear and nonlinear

Volterra integral equations. Volterra integral equations are essential for modeling time-dependent processes with past events. This method can be applied to solve the problems in a straightforward and simple manner without changing the physical behavior of the model. The role of the kernel function is clearly seen for solving these equations. MATLAB is used to distinguish the solution of each component and the exact solutions. This method can reduce the size of the computational work and spending time. Series solutions from each component converge to the exact solution by using Adomian decomposition method. Hence, using this method it can help us to solve the linear and nonlinear Volterra integral equation easily and simply.

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