Variational Iteration Method for Solving Linear Integral Equations of Second Kind

THEINT THINZAR AUNG[1], KHIN THAN SINT[2]
[1]Department of Mathematics, Mandalar University
[2]Department of Mathematics, University of Mandalay

Abstract- Firstly, the definition of integral equation and its concepts are introduced. Next, the variational iteration method is used for solving linear Fredholm and Volterra integral equations. The method provides rapidly convergent successive approximations of the exact solution. Finally, the variational iteration method is applied to find the solutions of the linear integral equations.

Keywords: Variational Iteration Method, Fredholm Integral Equations, Volterra Integral Equations.

I. INTRODUCTION

Joseph Fourier (1768-1830) is considered to be the initiator of the theory of integral equations and a statement of the solution to a first-kind integral equation. A Swedish mathematician, Ivar Fredholm (1866-1927), who developed the Fredholm integral equation and founded modern integral equation theory. Published a paper that produced the essential parts of what in 1990. An Italian mathematician, Vito Volterra, pioneered the systematic investigation of integral equations in the late 19th and early 20th century.

The variational iteration method (VIM) was developed by Chinese mathematician Ji-Huan He, Professor at Donghua University, in 1999. Initially proposed at the end of the most recent century and fully developed in 2006 and 2007. The variational iteration method was successfully applied to integral equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed-form solution exists [2].

In this paper, the variational iteration method, a newly developed method, is used for solving linear Fredholm integral equations and linear Volterra integral equations.

The paper is organized as follows. In section 1, the descriptions of the variational iteration method are presented with Fredholm integral equations and Volterra integral equations. In section 2, linear Fredholm integral equations are illustrated by using the variational iteration method. In section 3, linear Volterra integral equations are also calculated by using the variational iteration method. Examples will be examined to show the analysis and to confirm the efficiency of the method [2].

II. SOME CONCEPTS OF VIM

In section 1, definitions of integral equations are expressed by referring to Wazwaz, A. M, in 1996, 2011, and 2015. And then, the variational iteration method VIM and its basic concepts are presented.

Definition 1
An integral equation is an equation in which the unknown function $v(x)$ appears inside an integral sign. The most standard type of integral equation in the unknown function is of the form:

$$v(x) = f(x) + \lambda \int_{a(x)}^{b(x)} K(x,t)v(t)dt,$$

where $a(x)$ and $b(x)$ are the limits of integration, $K(x,t)$ is a known function of two variables x and t, and λ is a constant parameter, referred to as the kernel or nucleus of the integral equation [4].

In an integral equation, the unknown function may appear in one of two ways: (i) inside the integral sign, the function to be solved is only present within the integrand, and (ii) both inside and outside the integral sign, such as the unknown function appears both within the integral and as an external term. The functions $f(x)$ and $K(x,t)$ are given in advance. Integral equations appear in many forms.

Two distinct ways that depend on the limits of integration are used to characterize integral equations, namely:
(a) Fredholm integral equation,

(b) Volterra integral equation

Moreover, two other kinds that depend on the appearance of the unknown function $v(x)$ are defined as follows [2]:

(i) first kind Fredholm or Volterra integral equation,

(ii) second kind Fredholm or Volterra integral equation.

Definition 2

If the limits of integration are fixed, the integral equation is referred to as a Fredholm integral equation. If the unknown function $v(x)$ appears inside and outside the integral sign, this equation is called a Fredholm integral equation of the second kind. The form represents the second kind [2]:

$$v(x)=f(x)+\lambda\int_a^b K(x,t)v(t)dt.$$

Definition 3

If at least one limit is a variable, the equation is called a Volterra integral equation. However, the unknown function $v(x)$ appears inside and outside the integral sign. This is called the Volterra integral equation of the second kind. The form represents the second kind:

$$v(x)=f(x)+\lambda\int_0^x K(x,t)\,v(t)dt.$$

Definition 4

If the exponent of the unknown function $v(x)$ inside the integral sign is one, the integral equation or the integro-differential equation is called linear. Examples of linear Volterra integral equations and linear Fredholm integral equations are:

$$v(x)=5+2x^2-\int_0^x (x-t)v(t)dt,$$

and

$$v(x)=1+9x+2x^2+x^3-\int_0^1\left(20xt+10x^2t^2\right)v(t)dt,$$

respectively [4].

III.DESCRIPTIONS OF THE VARIATIONAL ITERATION METHOD

We consider the differential equation:

$$Ly+Ny=g(t), \qquad (1)$$

where the operator L is linear, whereas the operator N is nonlinear and $g(t)$ is the source inhomogeneous term. The variational iteration method presents a correction functional for equation (1) in the form [4]:

$$v_{n+1}(x)=v_n(x)+\int_0^x \lambda(\xi)\big(Lv_n(\xi)$$

$$+Nv_n(\xi)-g(\xi)\big)d\xi \qquad (2)$$

Where a general Lagrange multiplier λ may be a constant or a function, and v_n is a restricted value that means it behaves as a constant, hence $\delta v_n=0$, where δ is the variational derivative.

For the variational iteration method, two steps are expressed as follows, namely:

(i) the determination of the Lagrange multiplier that will be identified optimally, and

(ii) with λ determined, we substitute the result into (2), where the restrictions should be omitted.

Taking the variation of (2) for the independent variable v_n gives [2]

$$\frac{\delta v_{n+1}}{\delta v_n}=1+\frac{\delta}{\delta v_n}\left(\int_0^x \lambda(\xi)\big(Lv_n(\xi)+Nv_n(\xi)-g(\xi)\big)d\xi\right).$$

For the determination of the Lagrange multiplier, integration by parts is usually used. In other words, we can apply

$$\int_0^x \lambda(\xi)v_n'(\xi)d\xi=\lambda(\xi)v_n(\xi)-\int_0^x \lambda'(\xi)v_n(\xi)d\xi,$$

$$\int_0^x \lambda(\xi)v_n''(\xi)d\xi=\lambda(\xi)v_n'(\xi)-\lambda'(\xi)v_n(\xi)$$

$$+\int_0^x \lambda''(\xi)v_n(\xi)d\xi,$$

$$\int_0^x \lambda(\xi)v_n'''(\xi)d\xi=\lambda(\xi)v_n''(\xi)-\lambda'(\xi)v_n'(\xi)$$

$$+\lambda''(\xi)v_n(\xi)-\int_0^x \lambda'''(\xi)v_n(\xi)d\xi,$$

and so on. These identities are collected by integrating by parts [4].

Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximations v_{n+1}, $n\ge 0$, of the solution $v(x)$ will be readily obtained upon using the selective function $v_0(x)$. However, for fast convergence, the function $v_0(x)$ should be selected by using the initial conditions as follows:

$v_0(x) = v(0)$, for the first order v'_n,

$v_0(x) = v(0) + x\,v'(0)$, for the second order v''_n,

$v_0(x) = v(0) + x\,v'(0) + \frac{1}{2!}x^2 v''(0)$, for the third order v'''_n,

and so on. Consequently, the exact solution is

$$v(x) = \lim_{n\to\infty} v_n(x).$$

In other words, the correction functional (2) will give several approximations, and therefore, the exact solution is acquired as the limit of the resulting successive approximations [4].

IV. APPLICATIONS TO LINEAR FREDHOLM INTEGRAL EQUATIONS

In section 2, the solutions of linear Fredholm integral equations of the second kind are calculated by using the variational iteration method [3].

Illustrative Solution of Linear Fredholm Integral Equation with VIM

We consider the linear Fredholm integral equation

$$v(x) = e^x - x + x\int_0^1 t\,v(t)\,dt. \qquad (3)$$

Differentiating both sides of this equation with respect to x yields

$$v'(x) = e^x - 1 + \int_0^1 t\,v(t)\,dt.$$

The correction functional for this equation is given by

$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v'_n(\xi) - e^{\xi} + 1 - \int_0^1 \gamma v_n(\gamma)\,d\gamma \right) d\xi,$$

where we used $\lambda = -1$ for this equation. This initial condition $v(0) = 1$ is obtained by substituting $x = 0$ into (3).

We can use the initial condition to select $v_0(x) = v(0) = 1.$ Using this selection in the correction functional gives the following successive approximations:

$$v_0(x) = 1,$$

$$v_1(x) = v_0(x) - \int_0^x \left(v'_0(\xi) - e^{\xi} + 1 - \int_0^1 \gamma v_0(\gamma)\,d\gamma \right) d\xi$$

$$= 1 - \int_0^x \left(-e^{\xi} + 1 - \frac{1}{2} \right) d\xi,$$

$$v_1(x) = e^x - \frac{1}{2}x,$$

$$v_2(x) = v_1(x) - \int_0^x \left(v'_1(\xi) - e^{\xi} + 1 - \int_0^1 \gamma v_1(\gamma)\,d\gamma \right) d\xi$$

$$= v_1(x) - \int_0^x \left(-\frac{1}{2} + 1 - \int_0^1 \gamma \left(e^{\gamma} - \frac{1}{2}\gamma \right) d\gamma \right) d\xi$$

$$= e^x - \frac{1}{2\times 3}x,$$

$$v_3(x) = v_2(x) - \int_0^x \left(v'_2(\xi) - e^{\xi} + 1 - \int_0^1 \gamma v_2(\gamma)\,d\gamma \right) d\xi$$

$$= e^x - \frac{x}{2\times 3} - \int_0^x \left(-\frac{1}{2\times 3} + 1 - 1 + \frac{1}{2\times 3^2} \right) d\xi$$

$$= e^x - \frac{1}{2\times 3^2}x,$$

$$v_4(x) = v_3(x) - \int_0^x \left(v'_3(\xi) - e^{\xi} + 1 - \int_0^1 \gamma v_3(\gamma)\,d\gamma \right) d\xi_n$$

$$= e^x - \frac{1}{2\times 3^3}x,$$

and so on. This, in turn, gives

$$v_{n+1}(x) = e^x - \frac{1}{2\times 3^n}x, n \ge 0.$$

The variational iteration method admits the use of $v(x) = \lim_{n\to\infty} v_n(x)$ that gives the exact solution by

$$v(x) = e^x.$$

Solving Linear Fredholm Integral Equation with VIM

We consider the linear Fredholm integral equation

$$v(x) = \cos x + 2x + \int_0^{\pi} x\,t\,v(t)\,dt. \qquad (4)$$

Differentiating both sides of this equation (4) with respect to x yields

$$v'(x) = -\sin x + 2 + \int_0^{\pi} t\,v(t)\,dt. \qquad (5)$$

The correction functional for this equation (5) is given by

$$v_{n+1}(x) = v_n(x) - \int_0^x \Big(v_n'(\xi) + \sin\xi - 2 - \int_0^{\pi} \gamma\, v_n(\gamma)\, d\gamma \Big) d\xi,$$

where we used $\lambda = -1$ for this equation. This initial condition $v(0) = 1$ is obtained by substituting $x = 0$ into (4).

Using this selection in the correction functional gives the following successive approximations [4]:

$v_0(x) = 1,$

$$v_1(x) = v_0(x) - \int_0^x \left(v_0'(\xi) + \sin\xi - 2 - \int_0^{\pi} \gamma v_0(\gamma)\, d\gamma \right) d\xi$$

$$= v_0(x) - \int_0^x \left(\sin\xi - 2 - \int_0^{\pi} \gamma\, d\gamma \right) d\xi$$

$$= \cos x + \left(2x + \frac{\pi^2}{2} x \right),$$

$$v_2(x) = v_1(x) - \int_0^x \left(v_1'(\xi) + \sin\xi - 2 - \int_0^{\pi} \gamma\, v_1(\gamma)\, d\gamma \right) d\xi$$

$$= v_1(x) - \int_0^x \Bigg(-\sin\xi + 2 + \frac{\pi^2}{2} + \sin\xi - 2 - \int_0^{\pi} \gamma \left(\cos\gamma + 2\gamma + \frac{\pi^2}{2}\gamma \right) d\gamma \Bigg) d\xi$$

$$= \cos x + \left(2x + \frac{\pi^2}{2} x \right) + \left(-2x - \frac{\pi^2}{2} x + \frac{2}{3}\pi^3 x + \frac{\pi^5}{6} x \right)$$

$$= \cos x + \left(\frac{2}{3}\pi^3 x + \frac{\pi^5}{6} x \right),$$

$$v_3(x) = v_2(x) - \int_0^x \left(v_2'(\xi) + \sin\xi - 2 - \int_0^{\pi} \gamma\, v_2(\gamma)\, d\gamma \right) d\xi$$

$$v_3(x) = v_2(x) - \int_0^x \left(\frac{2}{3}\pi^3 + \frac{\pi^5}{6} - 2 + 2 - \frac{2}{9}\pi^6 - \frac{\pi^8}{18} \right) d\xi$$

$$= \cos x + \left(2x + \frac{\pi^2 x}{2} \right) + \left(-2x - \frac{\pi^2 x}{2} \right) + \frac{2\pi^3 x}{3} + \frac{\pi^5 x}{6} + \left(-\frac{2}{3}\pi^3 x - \frac{\pi^5}{6} x + \frac{2}{9}\pi^6 x + \frac{\pi^8}{18} x \right),$$

and so on. Canceling the noise terms, the exact solution is given by [4]

$$v(x) = \cos x.$$

V. APPLICATION TO THE LINEAR VOLTERRA INTEGRAL EQUATION

In section 3, the solution of the linear Volterra integral equation of the second kind is calculated by using the variational iteration method. Numerical results show that our proposed method has more accuracy and effect.

Solving Linear Volterra Integral Equation with VIM

We consider the linear Volterra integral equation [3]

$$v(x) = 1 + \int_0^x v(t)\, dt\,. \qquad (6)$$

Using the Leibniz rule to differentiate both sides of (6) with respect to x gives the equation

$$v'(x) = v(x)\,. \qquad (7)$$

Substituting $x = 0$ in (6) gives the initial condition $v(0) = 1$. Using the variational iteration method, the correction functional for (7) is

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi)\big(v_n'(\xi) - v_n(\xi) \big) d\xi.$$

(8)

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the correction functional (8) gives the iteration formula [4]

$$v_{n+1}(x) = v_n(x) - \int_0^x \big(v_n'(\xi) - v_n(\xi) \big) d\xi. \qquad (9)$$

We can use the initial condition to select $v_0(x) = v(0) = 1.$

Using this selected function in (9) gives the following successive approximations:

$v_0(x) = 1,$

$$v_1(x) = v_0(x) - \int_0^x \left(v_0'(\xi) - v_0(\xi)\right) d\xi$$

$$v_1(x) = 1 - \int_0^x (0-1)\, d\xi = 1 - \left[-(x-0)\right] = 1 + x,$$

$$v_2(x) = v_1(x) - \int_0^x \left(v_1'(\xi) - v_1(\xi)\right) d\xi$$

$$= 1 + x - \int_0^x \left(1 - (1+\xi)\right) d\xi = 1 + x + \frac{1}{2!}x^2,$$

$$v_3(x) = v_2(x) - \int_0^x \left(v_2'(\xi) - v_2(\xi)\right) d\xi$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3,$$

and so on,

$$v_9(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

$$+ \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}x^9,$$

and so on. This, in turn, gives

$$v_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5$$

$$+ \frac{1}{6!}x^6 + \ldots + \frac{1}{n!}x^n.$$

The variational iteration method [4] admits the use of

$$v(x) = \lim_{n\to\infty}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}\right),$$

that gives the exact solution $v(x) = e^x$.

Table 1: Numerical results of integral equation 6, N=10 (tenth iteration)

x	$v_{Exact}(x)$	v_{VIM}	Error ($v_{Exact} - v_{VIM}$)
0.1	1.105170918	1.105170918	--------
0.2	1.221402758	1.221402758	--------
0.3	1.349858808	1.349858808	--------
0.4	1.491824698	1.491824698	--------
0.5	1.648721271	1.648721270	1×10^{-9}
0.6	1.822118800	1.822118799	1×10^{-9}
0.7	2.013752707	2.013752699	8×10^{-9}
0.8	2.225540928	2.225540897	31×10^{-9}
0.9	2.459603111	2.459603007	104×10^{-9}
1.0	2.718281828	2.718281526	302×10^{-9}

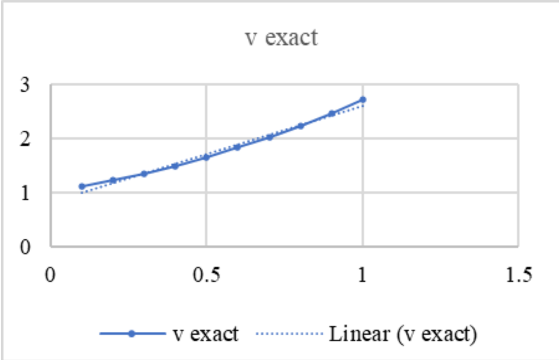

Figure 1: The exact solutions of the Volterra integral equation 6

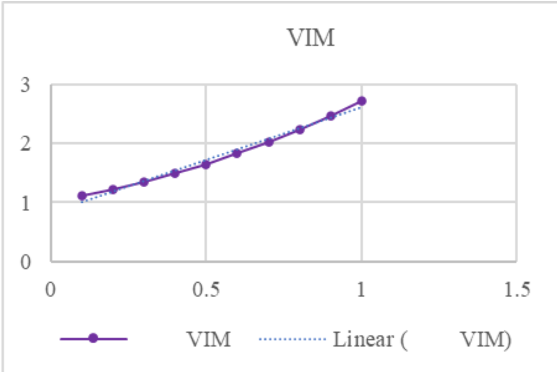

Figure 2: The approximate solutions of the Volterra integral equation 6

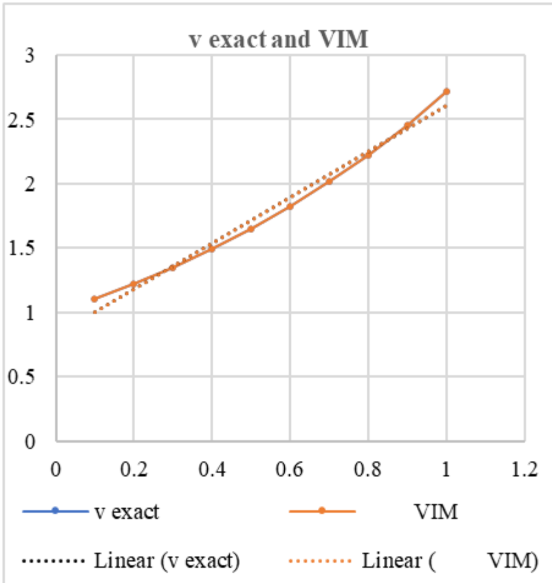

Figure 3: Comparison of the exact solutions and approximate solutions results of the Volterra integral equation 6

The graphical representations of the analytical solution and VIM-derived solutions are presented in

Figures 1 and 2, respectively. Figure 3 provides a visual comparison between the exact solutions and the approximate solutions generated through the variational iteration method. To evaluate the performance of the proposed method, we calculated the absolute errors $E_{10}(v)$ for this test case. Table 1 represents a systematic comparison between the numerical solutions obtained via VIM and the exact solutions for the specified Volterra integral equation 6.

The numerical results demonstrate strong convergence between the exact solution and the approximate solution generated by the proposed variational iteration method. The organized numerical data output includes the computed absolute errors, which serve to quantify the method's precision. The variational iteration method exhibits better performance metrics than other numerical methods.

CONCLUSION

In this paper, we construct the variational iteration method and use it to solve the linear Fredholm and Volterra integral equations. The variational iteration method (VIM) has proven to be an effective analytical tool for solving both linear and nonlinear integral equations. This study has successfully applied its methods to Fredholm and Volterra integral equations through numerical examples. The numerical experiments confirm VIM's reliability and effectiveness in handling integral equations, making it a valuable tool for both theoretical and applied mathematical analysis. Therefore, the proposed method is a very effective tool for calculating the exact solutions of integral equations.

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