© NOV 2025 | IRE Journals | Volume 9 Issue 5 | ISSN: 2456-8880

An Analytical Approach to Quadratic Optimal Control Problems Governed by Higher-Order Ordinary Differential Equations

AYODEJI SUNDAY AFOLABI
Department of Mathematical Sciences, Federal University of Technology, Akure.

Abstract- ***This research presents a comprehensive analytical framework for solving quadratic optimal control problems (OCPs) governed by higher-order ordinary differential equations (ODEs). The core methodology involves transforming the higher-order system into an equivalent first-order system through state variable expansion, thereby enabling the direct application of Pontryagin's Maximum Principle. This transformation facilitates the derivation of the necessary conditions for optimality, leading to a coupled system of state and costate equations. The general solution to this two-point boundary value problem is constructed analytically, utilizing matrix exponentials and eigenpair expansions for linear systems with constant coefficients. The efficacy of the analytical approach is demonstrated through detailed solutions to illustrative examples involving second-order ODE constraints. The results con- firm the method's ability to yield explicit expressions for the optimal state, control, and adjoint trajectories, establishing it as a robust and systematic analytical tool for this important class of OCPs.***

Keywords: ***Optimal Control, Higher-Order Odes, Pontryagin's Principle, Analytical Solutions, Adjoint Equations***

I. INTRODUCTION

Optimal control theory provides a powerful mathematical framework for determining control policies that optimize a given performance index while satisfying the dynamical constraints of the system. In most cases, this class of problems are formulated with system dynamics expressed as first-order differential equations that describe the rate of change of state variables with respect to time. However, many physical, biological, economical and engineering systems are governed by higher-order differential equations, where the system's evolution depends not only on the present state and control but also on higher-order derivatives. Examples include mechanical systems involving acceleration and jerk, beam and plate vibration models, and certain epidemiological or economic models characterized by delayed or cumulative effects [15–17].

The study of optimal control problems governed by higher-order differential equations extends the classical framework to accommodate these complex dynamics. In such problems, the control objective is governed by differential constraints of order greater than one. Several mathematical tools have been developed to analyze and solve higher- order control problems, notably the calculus of variations, Pontryagin's Maximum Principle (PMP) generalized to higher-order systems, and Hamiltonian formulations involving higher-order adjoint variables. These methods enable the derivation of necessary conditions for optimality that enable both theoretical analysis and numerical implementation. The growing importance of such systems is in their ability to accurately describe real- world processes that cannot be adequately captured by first-order formulations. Con- sequently, the investigation of higher-order optimal control systems not only enhances the theoretical foundation of control theory but also broadens its applicability to many disciplines [9, 16].

The quest for optimality is essential to the development of science and engineering. The mathematical field of optimization is fundamentally concerned with figuring out the greatest feasible result—minimizing effort or maximizing benefit—under specific constraints. It supports choices in scientific modeling, engineering design, and economics by locating the extrema of a function that represents system performance. Because feasible solutions may change, local minima may appear, and the global solution may be located on the edge of the feasible region, the introduction of constraints greatly complicates the search for an optimal solution. Thus, creating reliable algorithms for restricted optimization continues to be a major applied mathematics task [15, 16].

Many researchers have worked on optimal control problems constrained ordinay differential equations [4–8, 10–14, 18] . [9] provided a comprehensive introduction to optimizing dynamic biological systems, progressively building from basic theory to advanced applications including optimal control problem governed by higher-order differential equations. Pontryagin's Principle traditionally formulated for first-order systems was seamlessly incorporated for solving this class of problems by transforming it into an equivalent system of first-order equations through state variable expansion. This technique involves defining new state variables for each derivative, thereby allowing the standard framework of Hamiltonians and adjoints to be directly applied. This crucial step significantly broadens the scope of the method, ensuring that optimal control theory can be employed for a wide array of complex biological models beyond simple, first-order dynamics [16].

In [2, 3], the authors investigated optimal control problems (OCPs) governed by systems of linear ordinary differential equations (ODEs) subject to mixed constraints. The analytical solutions of these problems were derived by applying the first-order optimality conditions to the Hamiltonian and Lagrangian functions, respectively. In each case, the application of these conditions led to a system of first-order differential equations, which were solved using the method of the fundamental matrix. Through this approach, the optimal state, control, and adjoint variables, as well as the optimal value of the objective function, were explicitly determined. Furthermore, two illustrative examples were presented in each study to demonstrate the analytical procedure one for OCPs constrained solely by ordinary differential equations and another for OCPs involving both equality and inequality (mixed) constraints.

This paper focuses on the formulation, analysis, and solution techniques of optimal control problems governed by higher-order differential equations. Emphasis is placed on establishing the necessary conditions for optimality, exploring the equivalence between higher-order and first-order formulations through state transformations.

II. METHODOLOGY

In this section, optimal control problems (OCPs) constrained by higher-order ordinary differential equations (ODEs) with real coefficients are considered. The higher-order ODE is transformed into an equivalent system of first-order ODEs. The necessary conditions for optimality are then derived by applying the first-order optimality conditions to the Hamiltonian function. This approach yields the analytical solutions of the problem.

2.1 Necessary Conditions for an Optimal Control Problem with Higher-Order ODE Constraints

Consider the general higher-order optimal control problem of the form

$$\begin{aligned} \text{Minimize}\quad J(x,u) &= \int_{t_0}^{t_1} f\big(t, x(t), x'(t), \dots, x^{(n)}(t), u(t)\big)\,dt, \\ \text{subject to}\quad x^{(n+1)}(t) &= g\big(t, x(t), x'(t), \dots, x^{(n)}(t), u(t)\big), \\ x^{(k)}(t_0) &= a_{k+1}, \qquad k = 0, 1, \dots, n, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathrm{R}$ and $u(t) \in \mathrm{R}^m$. The functions f and g are continuously differentiable with respect to all their arguments.

To apply Pontryagin's Maximum Principle, we transform the higher-order system into a first-order system by introducing auxiliary state variables

$$x_1(t) = x(t), \quad x_2(t) = x'(t), \quad \dots, \quad x_{n+1}(t) = x^{(n)}(t).$$

Hence, the equivalent first-order system becomes

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= x_3(t), \\ &\;\vdots \\ x_n'(t) &= x_{n+1}(t), \\ x_{n+1}'(t) &= g\big(t, x_1, x_2, \dots, x_{n+1}, u\big), \end{aligned} \qquad x_i(t_0) = a_i, \;\; i = 1, \dots, n+1. \tag{2.2}$$

The Hamiltonian H for this problem is defined as:

$$H = f(t, x_1, \dots, x_{n+1}, u) + \sum_{i=1}^{n} \lambda_i x_{i+1} + \lambda_{n+1} g(t, x_1, \dots, x_{n+1}, u). \tag{2.3}$$

The adjoint variables $\lambda_1, \dots, \lambda_{n+1}$ satisfy the costate equations:

$$\dot{\lambda}_i(t) = -\frac{\partial H}{\partial x_i}, \quad \text{for } i = 1, \dots, n+1. \tag{2.4}$$

Expanded explicitly, these equations are:

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial f}{\partial x_1} - \lambda_{n+1}\frac{\partial g}{\partial x_1}, \\
\dot{\lambda}_2 &= -\frac{\partial f}{\partial x_2} - \lambda_1 - \lambda_{n+1}\frac{\partial g}{\partial x_2}, \\
\dot{\lambda}_3 &= -\frac{\partial f}{\partial x_3} - \lambda_2 - \lambda_{n+1}\frac{\partial g}{\partial x_3}, \\
&\vdots \\
\dot{\lambda}_{n+1} &= -\frac{\partial f}{\partial x_{n+1}} - \lambda_n - \lambda_{n+1}\frac{\partial g}{\partial x_{n+1}}.
\end{aligned}$$

The optimal control $u^*(t)$ must satisfy the necessary condition:

$$\frac{\partial H}{\partial u} = \frac{\partial f}{\partial u} + \lambda_{n+1}\frac{\partial g}{\partial u} = 0. \tag{2.5}$$

For the given initial conditions $x_i(t_0) = a_i$, $i = 1, \dots, n+1$, the terminal conditions for the adjoint variables are given by the transversality conditions. If the terminal state $x_i(t_1)$ is free, then

$$\lambda_i(t_1) = 0. \tag{2.6}$$

If a terminal cost $\Phi(x_1(t_1), \dots, x_{n+1}(t_1))$ is present in the objective functional, then

$$\lambda_i(t_1) = -\frac{\partial \Phi}{\partial x_i}(x^*(t_1)), \quad i = 1, \dots, n+1. \tag{2.7}$$

2.2 Analytical Framework for Higher-Order Optimal Control Problems

Consider the general higher-order optimal control problem of the form

$$\begin{aligned}
\text{Minimize} \quad J(x, u) &= \int_{t_0}^{t_1} f\big(t, x(t), x'(t), \dots, x^{(n)}(t), u(t)\big)\, dt, \\
\text{subject to} \quad x^{(n+1)}(t) &= g\big(t, x(t), x'(t), \dots, x^{(n)}(t), u(t)\big), \\
x^{(k)}(t_0) &= a_{k+1}, \quad k = 0, 1, \dots, n,
\end{aligned} \tag{2.8}$$

where $x(t) \in R$ and $u(t) \in R^m$. The functions f and g are assumed to be continuously differentiable with respect to all their arguments.

Theorem 2.1. *Let $x^*(t)$ and $u^*(t)$ be an optimal state-control pair that minimizes $J(x, u)$ subject to* (2.8)*. Then there exist adjoint variables $\lambda_0(t), \lambda_1(t), \dots, \lambda_n(t)$ such that the following necessary conditions hold:*

$$\dot{x}_k(t) = x_{k+1}(t), \quad k = 0, 1, \dots, n-1, \tag{2.9}$$

$$\dot{x}_n(t) = g\big(t, x_0(t), x_1(t), \dots, x_n(t), u(t)\big), \tag{2.10}$$

$$\dot{\lambda}_k(t) = -\frac{\partial H}{\partial x_k}, \quad k = 0, 1, \dots, n, \tag{2.11}$$

$$0 = \frac{\partial H}{\partial u}, \tag{2.12}$$

where the Hamiltonian function is defined as

$$H(t, x, u, \lambda) = f\left(t, x_0, x_1, \ldots, x_n, u\right) + \lambda_n\, g\left(t, x_0, x_1, \ldots, x_n, u\right) + \sum_{k=0}^{n-1} \lambda_k\, x_{k+1}. \qquad (2.13)$$

The terminal transversality conditions are given by

$$\lambda_k(t_1) = 0, \quad k = 0, 1, \ldots, n, \qquad (2.14)$$

if the terminal values of the corresponding state variables $x_k(t_1)$ are free.

Proof. We introduce auxiliary state variables to reduce the higher-order problem (2.8) to a first-order system by defining

$$x_0(t) = x(t), \quad x_1(t) = x'(t), \quad \ldots, \quad x_n(t) = x^{(n)}(t).$$

Then the dynamics can be written as

$$\dot{x}_k(t) = x_{k+1}(t), \quad k = 0, 1, \ldots, n-1, \qquad \dot{x}_n(t) = g\left(t, x_0, x_1, \ldots, x_n, u\right).$$

Applying the Pontryagin's Maximum Principle (PMP) to this first-order system yields the Hamiltonian (2.13) and the adjoint system (2.11). The stationarity condition (2.12) follows from the first-order necessary condition for optimality, ensuring that H is minimized with respect to u at each time t.

Equations (2.9)–(2.12) form a coupled system of $2(n+1)$ first-order differential equations, together with the appropriate boundary and transversality conditions. The optimal solution $(x^*(t), u^*(t))$ is obtained by simultaneously solving the state and costate equations under these constraints.

For the compact matrix representation, let

$$X(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad \Lambda(t) = \begin{bmatrix} \lambda_0(t) \\ \lambda_1(t) \\ \vdots \\ \lambda_n(t) \end{bmatrix},$$

then the complete system can be expressed in matrix form as

$$\begin{pmatrix} \dot{X}(t) \\ \dot{\Lambda}(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} X(t) \\ \Lambda(t) \end{pmatrix} + \begin{bmatrix} B\,u(t) \\ -\dfrac{\partial f}{\partial x} - \lambda_n \dfrac{\partial g}{\partial x} \end{bmatrix}, \qquad (2.15)$$

where A is the companion matrix associated with the higher-order derivative structure. The optimal control $u^*(t)$ satisfies

$$\frac{\partial H}{\partial u}\left(t, X^*, u^*, \Lambda^*\right) = 0.$$

This framework provides a generalized structure for deriving analytical or semi- analytical solutions for higher-order optimal control problems using reduction to first- order systems, adjoint equations, and the Pontryagin's Maximum Principle.

Hence, the general solution is

$$Z(t) = e^{Mt}Z(0) + \int_0^t e^{M(t-s)}R\,ds = \sum_{i=1} \kappa_i v_i e^{\eta_i t} + Z_p,$$

where (η_i, v_i) are the eigenpairs of M, $\{\kappa_i\}$ are constants determined by boundary conditions, and Z_p is a constant particular solution solving

$$MZ_p + R = 0.$$

Constant (particular) solution: Write $Z_p = (a, \beta, \gamma, \delta)^\top$ with $a = x_{1p}, \beta = x_{2p}, \gamma = \lambda_{1p}, \delta = \lambda_{2p}$. Setting the time derivatives to zero yields the algebraic system

$$\beta = 0, \tag{2.16}$$

$$a_0 a - \frac{b^2}{2q}\delta - \frac{db}{2q}\Lambda = 0, \tag{2.17}$$

$$-2pa - a_0\delta - \Lambda c_1 = 0, \tag{2.18}$$

$$-\gamma - a_1\delta - \Lambda c_2 = 0. \tag{2.19}$$

From (2.17) and (2.18) we eliminate a and obtain δ explicitly:

$$a = \frac{b^2\delta + db\Lambda}{2qa_0},$$

substitute into (2.18) to get

$$2p\frac{b^2\delta + db\Lambda}{2qa_0} = -a_0\delta - \Lambda c_1,$$

Hence,

$$\delta\left(pb^2 + qa_0^2\right) = -\Lambda\left(qa_0c_1 + pdb\right).$$

Therefore the particular adjoint component $\delta = \lambda_{2p}$ is

$$\delta = -\Lambda\,\frac{qa_0c_1 + pdb}{pb^2 + qa_0^2}. \tag{2.20}$$

Then

$$a = \frac{b^2\delta + db\Lambda}{2qa_0} = \Lambda\,\frac{-b^2(qa_0c_1 + pdb) + 2qa_0db}{2qa_0(pb^2 + qa_0^2)}, \tag{2.21}$$

and the other components are

$$\gamma = -a_1\delta - \Lambda c_2, \qquad \beta = 0. \tag{2.22}$$

Thus, the particular control is

$$u_p = -\frac{1}{2q}\left(b\,\delta + d\Lambda\right). \tag{2.23}$$

Applying the boundary conditions, the complete solution is obtained as

$$Z(t) = \sum_{i=1} \kappa_i v_i e^{\eta_i t} + Z_p,$$

and the constants $\{\kappa_i\}$ (equivalently the initial adjoint $\lambda(0)$ and homogeneous coefficients) are determined by the 4 boundary conditions: the two initial state conditions

$$x_1(0) = x_0, \qquad x_2(0) = v_0,$$

and the two terminal adjoint conditions (for free terminal state)

$$\lambda_1(T) = 0, \qquad \lambda_2(T) = 0.$$

Solving the resulting linear system fixes the κ_i. If a terminal cost $\Phi(X(T))$ is present, replace the terminal adjoint values by $\lambda(T)\,\nabla_x\Phi(X(T))$.

This completes the analytic construction for the second-order case, and the same procedures are applicable to problems where the system order is greater than two.

III. RESULTS AND DISCUSSIONS

This section demonstrates the application of the necessary conditions derived in Section 2.1 by presenting complete solutions to several optimal control problems constrained by higher-order differential equations.

Example 1. Problem

$$\min_u \frac{1}{2}\int_0^{\pi}\left(u(t)^2 - 0.5x(t)^2\right)dt \tag{3.1}$$

subject to

$$x''(t) = u(t) + x(t), \quad x(0) = 1, \quad x'(0) = 1. \tag{3.2}$$

Solution

Reformulating the state equation as a first-order system, we have Let $x_1 = x$, $x_2 = x'$. *The dynamics are:*

$$x_1' = x_2, \quad x_1(0) = 1,$$
$$x_2' = u + x_1, \quad x_2(0) = 1.$$

The Hamiltonian is given as:

$$H = \frac{1}{2}(u^2 - 0.5x_1^2) + \lambda_1 x_2 + \lambda_2(u + x_1).$$

Optimal control:

$$\frac{\partial H}{\partial u} = u + \lambda_2 = 0 \quad \Rightarrow \quad u^*(t) = -\lambda_2(t).$$

Adjoint equations:

$$\lambda_1' = -\frac{\partial H}{\partial x_1} = 0.5x_1 - \lambda_2, \quad \lambda_1(\pi) = 0,$$
$$\lambda_2' = -\frac{\partial H}{\partial x_2} = -\lambda_1, \quad \lambda_2(\pi) = 0.$$

The State and costate system are expressed We have:

$$x_1' = x_2,$$

$$x_2' = u + x_1 = -\lambda_2 + x_1,$$

$$\lambda_1' = 0.5x_1 - \lambda_2,$$

$$\lambda_2' = -\lambda_1.$$

Reducing the system of first-order ordinary differential equations above to a fourth-order ordinary differential equation, we obtain

$$x_1'' - x_1 = -\lambda_2$$

$$\lambda_2'' - \lambda_2 = -0.5x_1$$

$$x_1^{(4)} - 2x_1'' + 0.5x_1 = 0.$$

From the characteristic equation, let $r^4 - 2r^2 + 0.5 = 0$. *Let* $s = r^2$: $s^2 - 2s + 0.5 = 0$.

$$s = \frac{2 \pm \sqrt{4-2}}{2} = \frac{2 \pm \sqrt{2}}{2} = 1 \pm \frac{\sqrt{2}}{2}.$$

Numerically: $\sqrt{2}/2 \approx 0.707106781$, $s_1 \approx 1.707106781$, $s_2 \approx 0.292893219$.

Thus $r = \pm\sqrt{s_1} = \pm a$, $r = \pm\sqrt{s_2} = \pm b$, *where* $a \approx 1.306562965$, $b \approx 0.541196100$.

The general solution is given as

$$x_1(t) = Ae^{at} + Be^{-at} + Ce^{bt} + De^{-bt}.$$

Applying the Boundary conditions $x_1(0) = 1$, $x_2(0) = 1$, $\lambda_1(\pi) = 0$, $\lambda_2(\pi) = 0$, *we have*

$$\begin{aligned} A + B + C + D &= 1, \\ aA - aB + bC - bD &= 1, \\ -kAe^{at} - kBe^{-at} + kCe^{bt} + kDe^{-bt} &= \lambda_2(t), \\ kaAe^{at} - kaBe^{-at} - kbCe^{bt} + kbDe^{-bt} &= \lambda_1(t). \end{aligned} \tag{3.3}$$

Give the tTerminal conditions $\lambda_1(\pi) = 0$ *and* $\lambda_2(\pi) = 0$, *we obtain*

$$\begin{aligned} aAe^{a\pi} - aBe^{-a\pi} - bCe^{b\pi} + bDe^{-b\pi} &= 0, \\ -Ae^{a\pi} - Be^{-a\pi} + Ce^{b\pi} + De^{-b\pi} &= 0. \end{aligned} \tag{3.4}$$

Let $a \approx 1.306562965$, $b \approx 0.541196100$, $e^{a\pi} \approx e^{4.104} \approx 60.59$, $e^{-a\pi} \approx 0.01650$, $e^{b\pi} \approx e^{1.700} \approx 5.473$, $e^{-b\pi} \approx 0.1827$. *Hence, Hence,*

$$x^*(t) = Ae^{at} + Be^{-at} + Ce^{bt} + De^{-bt},$$

$$u^*(t) = 0.7071\left[Ae^{at} + Be^{-at} - Ce^{bt} - De^{-bt}\right],$$

Recall that

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1.306563 & -1.306563 & 0.541196 & -0.541196 \\ 79.14 & -0.02156 & -2.962 & 0.09888 \\ -60.59 & -0.01650 & 5.473 & 0.1827 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.5}$$

The solution of the 4×4 *linear system gives*

$$A = -0.018, \quad B = -2.545, \quad C = -0.3445, \quad D = 3.9075.$$

Thus, the optimal state and control variables are

$$\begin{aligned} x^*(t) &= -0.018\, e^{1.306563t} - 2.545\, e^{-1.306563t} - 0.3445\, e^{0.541196t} + 3.9075\, e^{-0.541196t}, \\ u^*(t) &= 0.7071 \left(-0.018\, e^{1.306563t} - 2.545\, e^{-1.306563t} + 0.3445\, e^{0.541196t} - 3.9075\, e^{-0.541196t} \right) \end{aligned}$$

Problem 2

Solve the optimal control problem:

$$\min_u \frac{1}{2} \int_0^{\pi} \left[5x(t)^2 + 3u(t)^2 \right] dt$$

subject to

$$x''(t) = x(t) + u(t), \quad x(0) = 1, \quad x'(0) = 1.$$

Solution

By expressing the state equation as a system of first-order equations, we define $x_1 = x$ and $x_2 = x'$. The resulting dynamics are given by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + u \end{cases} \quad \text{with initial conditions:} \quad x_1(0) = 1, \quad x_2(0) = 1.$$

The Hamiltonian is given as

$$\mathrm{H} = \frac{1}{2}(5x_1^2 + 3u^2) + \lambda_1 x_2 + \lambda_2(x_1 + u).$$

the optimality conditions is expressed as

$$\frac{\partial \mathrm{H}}{\partial u} = 3u + \lambda_2 = 0 \quad \Rightarrow \quad u^*(t) = -\frac{\lambda_2(t)}{3}.$$

and the costate equations are

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial \mathrm{H}}{\partial x_1} = -5x_1 - \lambda_2 \\ \dot{\lambda}_2 &= -\frac{\partial \mathrm{H}}{\partial x_2} = -\lambda_1 \end{aligned} \quad \text{with terminal conditions:} \quad \lambda_1(\pi) = 0, \quad \lambda_2(\pi) = 0.$$

Substituting $u = -\lambda_2/3$, we have

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - \dfrac{\lambda_2}{3} \\ \dot{\lambda}_1 = -5x_1 - \lambda_2 \\ \dot{\lambda}_2 = -\lambda_1 \end{cases} \quad \text{with boundary conditions:} \quad \begin{cases} x_1(0) = 1, \quad x_2(0) = 1 \\ \lambda_1(\pi) = 0, \quad \lambda_2(\pi) = 0 \end{cases}$$

Using matrix notation gives

$$\dot{X}(t) = MX(t) \tag{3.6}$$

where state-costate vector:

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} \text{ and } M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -\frac{1}{3} \\ -5 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The eigenvalues are obtained by applying the characteristic equation as follows:

$$\det(\lambda I - M) = 0 \quad \Rightarrow \quad (\lambda^2 - 1)^2 + \frac{5}{3} = 0$$

Let $k = \sqrt{\frac{5}{3}} \approx 1.290994$, then: $\lambda^2 = 1 \pm ik$.

Write $1 + ik = re^{i\theta}$ where:

$$r = \sqrt{1 + k^2} = \sqrt{\frac{8}{3}} \approx 1.632993, \quad \theta = \arctan(k) \approx 0.911738$$

Hence the eigenvalues are

$$\lambda = \pm\sqrt{r}e^{\pm i\theta/2} = \pm(\alpha \pm i\beta)$$

where

$$\alpha = \sqrt{r}\cos(\theta/2) \approx 1.1474, \quad \beta = \sqrt{r}\sin(\theta/2) \approx 0.5626$$

The general solution is of the form:

$$\begin{aligned} x_1(t) &= a_1 e^{\alpha t}\cos(\beta t) + a_2 e^{\alpha t}\sin(\beta t) + a_3 e^{-\alpha t}\cos(\beta t) + a_4 e^{-\alpha t}\sin(\beta t) \\ x_2(t) &= \dot{x}_1(t) \\ \lambda_2(t) &= b_1 e^{\alpha t}\cos(\beta t) + b_2 e^{\alpha t}\sin(\beta t) + b_3 e^{-\alpha t}\cos(\beta t) + b_4 e^{-\alpha t}\sin(\beta t) \\ \lambda_1(t) &= -\dot{\lambda}_2(t) \end{aligned}$$

Using the four boundary conditions: $x_1(0) = 1$, $x_2(0) = 1$, $\lambda_1(\pi) = 0$, $\lambda_2(\pi) = 0$,, initial costates: $\lambda_1(0) \approx 19.1016$, $\lambda_2(0) \approx 14.7733$, and

state coefficients: $\begin{matrix} a_1 \approx 0.00175, & a_2 \approx -0.00228 \\ a_3 \approx 0.9982, & a_4 \approx 3.8122 \end{matrix}$

Costate coefficients for $\lambda_2(t)$: $\begin{matrix} b_1 \approx 0.00883, & b_2 \approx 0.00679 \\ b_3 \approx 14.7644, & b_4 \approx -3.8662 \end{matrix}$.

Hence, the optimal state and control variables are

$$\begin{aligned} x^*(t) = {} & 0.00175e^{1.147t}\cos(0.563t) - 0.00228e^{1.147t}\sin(0.563t) \\ & + 0.9982e^{-1.147t}\cos(0.563t) + 3.8122e^{-1.147t}\sin(0.563t) \end{aligned}$$

$$\begin{aligned} u^*(t) = -\frac{\lambda_2(t)}{3} \approx -\frac{1}{3}\Big(& 0.00883e^{1.147t}\cos(0.563t) + 0.00679e^{1.147t}\sin(0.563t) \\ & + 14.7644e^{-1.147t}\cos(0.563t) - 3.8662e^{-1.147t}\sin(0.563t)\Big). \end{aligned}$$

IV. CONCLUSION

This study has successfully developed and validated a systematic analytical approach for solving quadratic optimal control problems constrained by higher-order ordinary differential equations. By leveraging a state variable transformation to reduce the problem to a first-order system, the powerful machinery of Pontryagin's Maximum Principle was directly applied to derive the necessary conditions for optimality. The resulting two-point boundary value problem was solved analytically, with the general solution expressed in terms of matrix exponentials and eigenpair expansions for constant-coefficient linear systems.

The practical application of this framework was demonstrated through complete analytical solutions to two numerical examples, showcasing the method's ability to determine explicit expressions for the optimal state and control variables. The procedure proves to be both rigorous and computationally tractable, providing a clear pathway from problem formulation to final solution. This research thus establishes a robust and unified analytical framework for addressing optimal control problems governed by higher-order dynamics, enhancing the theoretical toolkit available for such systems and offering a benchmark for validating numerical schemes.

REFERENCES

[1] Adebayo, K.J., Aderibigbe, F.A. and Akinmuyise, M.F. (2014), *On Application Of Modified Gradient to Extended Conjugate Gradient Method Algorithm For Solving Optimal Control Problems*. IOSR Journal of Mathematics, 9(5), 30–35.

[2] Afolabi, A.S. and Olotu, O. (2020), *Analytical Solutions of Optimal Control Problems with Mixed Constraints*. Iconic Research and Engineering Journals, 3(12), 1–9, India.

[3] Afolabi, A.S. (2022), *On the Constrained Optimal Control Problems with Mixed Constraints*. International Journal of Mathematical Analysis and Modelling, 5(2), 25–35, India.

[4] Afolabi, A.S. and Oyewale, A.A. (2025), *On the Solutions of Optimal Control Problems Constrained by Ordinary Differential Equations with Vector-Matrix Coefficients Using FICO Xpress Mosel*. Journal of Mathematical Sciences and Optimization: Theory and Application, 11(2), 111–132, Nigeria.

[5] Crankson, M.V., Afolabi, A.S. and Eshun, S. (2023), *Comprehensive Semi-Analytical Approach to Quadratic Optimal Control Problems Governed by Ordinary Differential Equations*. International Journal of Science and Research, 7(942), 2319–7064, India.

[6] Izmailov, A.F., Solodov, M.V. and Uskov, E.I. (2012), *Global Convergence of Augmented Lagrangian Methods Applied to Optimization Problems with Degenerate Constraints, Including Problems with Complementarity Constraints*. SIAM Journal on Optimization, 22(4), 1579–1606.

[7] Kahuru, J., Luboobi, L.S. and Gyekye, Y.N. (2017), *Optimal Control Techniques on a Mathematical Model for the Dynamics of Tungiasis in a Community*. International Journal of Mathematics and Mathematical Sciences, 2017, Article ID 4804897, 1–19.

[8] Lasdon, L.S., Mitter, S.K. and Warren, A.D. (1967), *The Conjugate Gradient Method for Optimal Control Problems*. IEEE Transactions on Automatic Control, 12(2), 132– 138.

[9] Lenhart, S., and Workman, J. T. (2007), *Lenhart, S., and Workman, J. T.*. ptimal Control Applied to Biological Models, Chapman and Hall/CRC, 12(2), 1–257.

[10] Olotu, O. (2011), *Discretized Scheme Examining Convergence and Geometric Ratio Profiles for Control Problem Constrained by Evolution Equation*. Journal of Mathematics and Statistics, 7(2), 116–123.

[11] Olotu, O. and Afolabi, A.S. (2014), *A New Scheme Using Penalty Functions with B-Spline Function for Constrained Optimal Control Problems*. International Journal of Numerical Mathematics (IJNM), 7(2), 282–300.

[12] Olotu, O. and Afolabi, A.S. (2017), *On the Optimal Control Problems Constrained by Delay Differential Equation Using Penalty Functions with B-Spline Function*. Journal of Mathematical Physics, Benin, Nigeria, 39, 119–128.

[13] Orji, J.C., Miswanto, and Afolabi, A.S. (2024), *Optimal Control Problems Con- strained by Ordinary Differential Equations Using Conjugate Gradient Method and FICO Xpress Mosel*. International Research and Innovation Journal, 1(1), 134–149, UK.

[14] Oyewale, A.A., Afolabi, A.S. and Dawodu, K.A. (2024), *On the Optimal Control Problems Constrained by Ordinary Differential Equations Using Conjugate Gradient Method and FICO Xpress Mosel*. Journal of Mathematics and Computational Science, 14(18), 1–25, UK.

[15] Polak, E. (1973), *An Historical Survey of Computational Methods in Optimal Control*. SIAM Review, 15(2), 552–584.

[16] Rao, S.S. (2019), *Engineering Optimization: Theory and Practice*, 5th Edition. John Wiley

and Sons, Hoboken, NJ, USA.

[17] Sofia, H. (2014), *Optimal Control and Numerical Optimization Applied to Epidemiological Models*. Ph.D. Thesis, Department of Mathematics, University of Aveiro, pp. 1–111.

[18] Zboon, R.A., Yadav, S.P. and Mohan, C. (1999), *Penalty Method for an Optimal Control Problem with Equality and Inequality Constraints*. Indian Journal of Pure and Applied Mathematics, 30(1), 1–14.