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Topology Construction on the Set of Non-Deranged Permutations Embedded in a Banach Space

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Abstract- This work improves on the embedding of groups into Banach space by first reconstructing the group of non deranged permutations into a vector space. We embed the space into a Banach space and define a topology on it. In the Banach space Rn there are many coordinates with different points and some of them are permutations of one or more coordinate points. By embedding a set of non deranged permutations in a Banach space, we seek to understand the set of non deranged permutations in light of the properties of a Banach space. Since Banach space was created to solve problems because a sequence of approximate solutions gets its limit in the Banach space, embedding the set of non deranged permutations in a Banach space makes it possible that a solution to an equation could come out in different permutation patterns. We use isometric embedding method and define a topology on the set. The structure of a topology ensures that the elements of the set can be comparable and relate to one other. The cycles are embedded into each other.

KEYWORDS: non deranged permutations, banach space, topology, embedding

STATEMENT OF NOVELTY:
Non deranged permutations is embedded into a Banach space. By the embedding, a solution to an equation can be in the form of different permutation pattern. By defining a topology on the set of non deranged permutations, the relationship between the elements is possible since topology structure on a set helps us to see how every element relates to each other. A norm defined on the set of non deranged permutations induces metrics and the metrics induces topology. Therefore the set of non deranged permutations is a topological space.

I. INTRODUCTION

The space of non-deranged permutations is an algebraic structure that has been the subject of several writers' works. Initially, Garba and Abubakar (2015) created an algebraic structure using modular arithmetic. For each, $Gp=\{w_1,w_2\cdots w_{p-1}\}$

Where $w_i = ((1)(1 + i)_{mp}(1 + 2i)_{mp}(1 + 3i)_{mp}\ldots(1 + (p-1)i)_{mp})$

The components of each w_i are unique and referred to as successors, and each w_i is referred to as a cycle. The algebraic structure G_p was embedded with a unique cycle, $w_p = \{(p)\}$. The structure that resulted was described as $G'p = G'p \cup \{w_p\}$

A concatenation map was defined as follows: $\varphi : G'p \times G'p \to G'p$ where $1 \le i,j \le p$, $p \ge 5$ is prime. Then (G'_p, φ) is an abelian group. Permutation was first used in China before it spread to other parts of the world. In Greece, Xenocrates discovered different syllables possible in Greek language using permutation. Permutation patterns have been used for decades to study mathematical structures. Permutation of a finite set is a bijective relation from the set to itself, McCoy (1968) and Nering (1970). A bijective relation refers to one-one mapping. If the number of elements in the set is n, then n is also called the degree of the permutation. A set's permutation group is the collection of every permutation that may be made from the set. The symmetric group of the set is the collection of all permutations of the set. The ordered configurations of a few chosen items from the set are known as partial permutations. There are n factorial ways to arrange n different objects.

Permutations are useful in many areas of mathematics and fields of science. Permutation is important in combinatorics and graph theory. In computer science, permutations are used in analyzing some algorithms. Permutations are used in many counting problems. According to Ibrahim (2007). In order to produce certain recurrent relations and functions for counting the number of subwords that result from the patterns, permutation patterns have been researched. Several prime generating functions

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were used to accomplish this. Based on the length of the developing subwords, a thorough counting scheme was developed for the patterns.Some group and graph theoretic consequences of these patterns were identified (Ibrahim, 2007). Ibrahim and Audu (2010) claim that these significant permutations demonstrate the non-commutative and non-associative characteristics of the Aunu permutation patterns' (123) avoidance patterns.The properties were established using Cayley table. Their results generated larger matrices from permutation of points of AUNU patterns of prime cardinality. Ibrahim and Audu (2010) noted that the generated symbols can be used in cryptography and game theory. Topology originates many centuries ago, Croom(1989). People such as Cauchy and Riemann made contributions in the field of topology, Richeson (2008). Embedding is a one to one structure preserving map. In category theory, the map is called a morphism. Several different embeddings of a set into another set are possible. Embedding is important because if there is a set X whose properties are not well known and there is a set Y whose properties are well known. Embedding X into Y helps to study the properties of X. There is standard embedding such as the natural numbers embedded in the integers and the real numbers embedded in the complex numbers. In topology, an embedding is a homeomorphism onto its image, Sharpe (1997). Embedding allows two spaces to be distinguished. If space X can be embedded into space Y while space Z cannot be embedded into space Y, then spaces X and Z are distinguishable. In field theory, an embedding of a field in another field is a ring homomorphism the kernel of which is an ideal of the codomain. In metric spaces, an embedding with distortion C is a map f:X, → Y which satisfies the condition $Ld_1(x,y) \leq d_2(f(x),f(y)) \leq CLd_1(x,y)$, ∀ x,y ∈ X. If X and Y are normed spaces, then the embedding satisfies $L\|x\| \leq \|y\| \leq CL\|x\|$ for any x ∈ X mapped to y ∈ Y, Nash (1956) and Piotr (2005). There is isometric embedding of spaces. Holystyski (1978) embedded l_∞ into l_1. Johnson and Schectan (1982) embedded l^m_p into l^n_1 where m < n. Johnson and Lidenstrauss (1989) proved a result on nearly isometric embeddings of infinite subsets of Euclidean spaces into low dimensional Euclidean space. Keith (1990) worked on embedding l_p where $1 < p \leq 2$. It was demonstrated by Deutrieux and Lancien (2000) that "any separable Banach space is linearly isometric to a subspace of Y whenever K embeds isometrically into a Banach space Y." Gyorgy, Tamas, and Daniel (2019) researched on isometric Wasserstein space embeddings.

Fractal embedding into Banach, Hilbert, and Euclidean spaces was the focus of Taras, Magdalene, and Flip's (2020) research. The coarse embedding of spaces is another. The limit set of the image of a coarse embedding of a geodesic space into a hyperbolic space of constrained geometry was examined by Brian (2019).

II. MATERIAL AND METHOD

0.2.1 Definition of some basic terms

Non deranged permutation

A non-deranged permutation is a permutation that has a fixed point.

Example:

i. The permutation of the elements (1234567) given by (1357246), (1473625), (1526374) , (1642753), (1765432), etc is non deranged because 1 is fixed.
ii. The permutation of the elements (abcdefg) given by (acegbfd), (dagcefb), (ebagcfd), etc is non deranged because f is fixed.

Deranged permutation
A deranged permutation is a permutation in which no element appears in its original position.

Example:

i. The permutation of the elements (1234567) given by (3572461), (4731625), (5126374) , (6412753), (7615432), etc is a deranged permutation because no number is fixed.
ii. The permutation of the elements (abcdefg) given by (fcegbad), (dfagceb), (ebafgcd), etc is deranged.

Vector space
Vector space: A vector space V over a field F is a set of elements with two defined operations called

addition and scalar multiplication such that the following properties hold:

i. Closure of addition: $\forall$ x, y $\in$ V, x + y $\in$ V
ii. Closure of scalar multiplication: $\forall$ α $\in$ F and $\forall$ x $\in$ V, αx $\in$ V
iii. Commutativity of addition: $\forall$ x, y $\in$ V, x + y = y + x
iv. associativity of addition: $\forall$ x, y, z $\in$ V, (x + y) + z = x + (y + z)
v. additive identity: $\exists$ 0 $\in$ V such that $\forall$ x $\in$ V, 0 + x = x
vi. additive inverse: $\forall$ x $\in$ V $\exists$ -x $\in$ V such that -x + x = 0.
vii. scalar identity: $\exists$ 1 $\in$ F such that $\forall$ x $\in$ V, 1x = x
viii. associativity of scalar with respect to field multiplication: $\forall$ α, β $\in$ F and x $\in$ V, (αβ) x = α(βx)m
ix. distributivity of scalar with respect to vector addition: $\forall$ x, y $\in$ V and α $\in$ F, α(x + y)= αx + αy
x. distributivity of scalar with respect to field addition: $\forall$ x $\in$ V and α, β $\in$ F, (α + β)x =αx + βx

Examples:

i. Let F be a field and n a natural number. Then Fn forms a vector space under tuple addition and scalar multiplication where the scalars are elements of F.
ii. Let X = B[a, b] be the set of all bounded real-valued functions on [a, b]. Then X is a vector space over R.Using the Concatenation map to build a mathematical structure

Garba and Abubakar (2015) utilized the concept of modular arithmetic to build a structure $G_p = \{w_1, w_2 \cdots w_{p-1}\}$ where each

$w_i = ((1)(1 + i)_{mp}(1 + 2i)_{mp}(1 + 3i)_{mp} \ldots (1 + (p-1)i)_{mp})$.

It is referred to as a cycle, and its constituent parts are unique and known as successors. The integers are taken modulo p, $1 \leq n \leq p$, as shown by the subscript mp. The structure's algebraic and number-theoretic characteristics were listed. π (w):=| △1f (w) |, where △1f (w) is the difference between the first and last successors in a cycle w, was used to define the length of each cycle. An operator $\pi: G_p \rightarrow X$ was also defined. An isomorphism on G_p is defined by this operator.

0.2.2 Extending G_p

The algebraic structure G_p was embedded with a special cycle wp = {(p)(p)(p)...(p)}, where the successors are not distinct and the special cycle has length 1. Garba and Abubakar (2015) defined the final structure as

0.2.3 G'p as an Abelian Group

φ_{ij}: $G'p \rightarrow G'p \times G'p$

where $1 \leq i,j \leq p, p \geq 5$ is prime. Then ($G'p$, φ) is an abelian group, Garba and Abubakar
(2015).

0.2.4 Γ_1-non-Deranged Permutation Group $G\,\Gamma_{1\ p}$

A Γ1-non deranged permutation group is defined by Garba, Ojonugwa, Aremu, and Hamisu (2017) as the permutation group that has a fixed element on the first column from the left.
For any prime p > 5, $1 \leq i \leq (p - 1)$.

0.2.5 Theorem

Let $G'p^*$ be a space of non deranged permutations taken over a Galois field Fp such that

$+: G'p* \times G'p* \rightarrow G'p^*$

And

$X: F_p \times G'p* \rightarrow G'p^*$

where + and × are vector addition and scalar multiplication signs respectively. Then $G'p^*$ is a vector space.

Proof. $w_i, w_j, w_k \in G'p^*$

i. Closure of vector addition : $w_i + w_j = w_{i+j} \in G'p^*$, because the addition is taken mod p.

For example: w_2= {1357246} and w_3= {1473625} in G_7. $w_2 + w_3 = w_5$ = {1642753}

ii. Closure of scalar multiplication: ∀ $w_i \in G'p^*$ and ∀ $a \in F_p$, $aw_i = w_{ai} \in G'p^*$

iii. Commutativity of addition: $w_i + w_j = w_{i+j} = w_{j+i} = w_j + w_i$

iv. Associativity of addition: $(w_i + w_j) + w_k = w_{(i+j)+k} = w_{i+(j+k)} = w_i + (w_j + w_k)$

v. Additive identity: wp is the additive identity since ∀ $w_i \in G'p^*$, $w_i + w_p = w_{i+p} = w_i$

vi. Additive inverse: Every w_i has an inverse in G'^*_p given by w_{p-i} because $w_i + w_{p-i} = wi{+}p{-}i = wp$

vii. Scalar identity: There exist $1 \in F_p$ such that $1w_p = p$

viii. Distributivity: ∀ $a \in F_p$ and w_i and $w_j \in G'p^*$, $a(w_i + w_j) = a(w_{i+j}) = w_{a(i+j)} = w_{ai+aj} = w_{ai} + w_{aj} = aw_i + aw_j$
$= wai + waj = awi + awj$

ix. Distributivity: ∀ a, b $\in F_p$ and $w_i \in G'p^*$, $(a+b)w_i = w_{(a+b)i} = w_{ai+bi} = w_{ai} + w_{bi} = aw_i + bw_i$

0.2.6 Theorem

There exists a subspace. $\{w_p\}$ of $G'p^*$

Proof. {wp} is closed with respect to vector addition and scalar multiplication by elements of Fp.

Closure of vector addition: $w_p + w_p = w_{2p} = w$, because 2p(mod p) = p

Closure of scalar multiplication: $\forall\ \alpha \in F_p$, $\alpha w_p = w_{\alpha p} = w_p$ because αp(mod p) = p.

0.2.7 Theorem

$\{w_1\}$is the basis for $G'p^*$

Proof. For all $w_i \in G'p^*$, $w_i = aw_1 = w_a$ where i = a $\in F_p$.

i. The cycle $w_3 \in G'7^*$, $w_3 = 3w_1$

ii. The cycle $w_4 \in G'7^*$, $w_4 = 4w_1$

$G'p^*$ has a demension 1 because there is only one element in the basis {w1}.

0.2.8 Theorem

Every vector in $G'p^*$ is linearly dependent.

Proof. Add every element of as follows:

$(w_1 + w_{p-1}) + (w_2 + w_{p-2}) + \ldots + (w_{p-1/2} + w_{p+1/2}) + w_p = w_p$

G'_p is linearly dependent.

III. RESULTS AND DISCUSSION

Embedding the vector space $G'p^*$ of non deranged permutations in the Banach space R^n

0.3.1 Theorem

A set of non deranged permutations can be embedded isometrically into a euclidean space.

Proof. Consider 2 elements wi and wj in G'p*. Let $d(w_i, w_j) = (\sum |wik - wjk|^2)^{1/2}$ where k is a natural number and w_{ik} is the k number in the cycle w_i. This is an isometric embedding of the group of permutations into R^n endowed with the l_2 norm.

Example Consider the vectors (1,3,5,7, 9, 11,2,4,6,8,10) and (1,8,4,11,7,3,10,6,2,9,5) in R^{11}. The l_2 distance is given by

$|1-1|^2+|3-8|^2+|5-4|^2+|7-11|^2+|9-7|^2+|11-3|^2+|2-10|^2+|4-6|^2+|6-2|^2+|8-9|^2+|10-5|^2$
$= |0|^2 + |-5|^2 + |1|^2 + |-4|^2 + |2|^2 + |8|^2 + |-8|^2 + |-2|^2 + |2|^2 + |-1|^2 + |5|^2$
$= 0 + 5^2 + 1^2 + 4^2 + 2^2 + 8^2 + 8^2 + 2^2 + 2^2 + 1^2 + 5^2$
$= 0 + 25 + 1 + 16 + 4 + 64 + 64 + 4 + 4 + 1 + 25 = 208$

We can see that w_2 = (1,3,5,7, 9,11,2,4,6,8,10) and w_7 = (1,8,4,11,7,3,10,6,2,9,5) in $G'p^*$ where p = 11.

Therefore there is an embedding of $G'p^*$ in the Banach space R^n

Definition of Topology on G'p*

0.3.2 Theorem

Let P($G'p$*) be the power set of $G_p'^*$. Then P(G'p*) forms a discrete topology on $G'p$*.

Proof. By the definition of power set, P(G'p*) contains

i. The empty set
ii. The set G'p*
iii. The union of subsets
iv. The intersection of the subsets The above are conditions which a topology must satisfied. Therefore P(G'p*)) is a discrete topology on G'p*.

0.3.3 Theorem

Define a norm $\|.\|$ on G'p*as

$\|.\|$: G'p* → F

$\|w_i\| = |i|$ for all w_i in $G'p$

Proof.

i. $\|w_i\| > 0$ since $\|w_i\| = |i| > 0$
ii. $\| aw_i \| = \| w_{ia} \| = | ia | = | a \| i | = | a | \| w_i \|$
iii. $\| w_i + w_j \| = \| w_{i+j} \| = | i + j | \leq | i | + | j | = \| w_i \| + \| w_j \|$ Then (G'p* $\|.\|$) is a normed vector space.

Norm induces metric and metric induces topology. By the definition of norm $\|.\|$ on G'p*, (G'p* $\|.\|$)) is a topological vector space.

Example:

i. $\|w_2\| > 0$ since $\|w_2\| = |2| > 0$
ii. $\| 4w_2 \| = \| w_{4\times 2} \| = | 4 \times 2 | = | 4 \| 2 | = | 4 | \| w_2 \|$
iii. $\| w2 + w3 \| = \| w2{+}3 \| = | 2 + 3 | \leq | 2 | + | 3 | = \| w2 \| + \| w3 \|$

0.3.4 Embedding of cycles

By defining a topology on the set G'p* of non deranged permutations, the elements of the set are comparable. For example consider w_2 = (1,3,5,7,9,11,2,4,6,8,10) and w_6 = (1,7,2,8,3,9,4,10,5,11,6) in G'11*, $3w_2 = w_6$. Each element of G'p* is a circle. For example: w_2= {1357246} and w_3= {1473625} in G_7. The difference between each member of w_2 and its successor is 2, ie 3-1=5-3=7-5=4-2=6-4=2, and in w_3, the difference between each member and its successor is 3. w_2= {1357246} is a circle with points 1,3,5,7,2,4,6 lying on it with space 2 between them, and w_3= {1473625} is a circle with points 1,4,7,3,6,2,5 lying on it with space 3 between them. These two circles are comparable because w_2= {1357246} can be embedded into the circle w_3= {1473625}. In each $G'p$*, w_i can be embedded into w_j with $i < j$.

IV. CONCLUSION

We make the following conclusions:

i. Non deranged permutations is embedded into a Banach space.
ii. By the embedding, a solution to an equation can be in the form of different permutation pattern. iii. By defining a topology on the set of non deranged permutations, the relationship between the elements is possible since topology structure on a set helps us to see how every element relates to each other.

V. RECOMMENDATION

The work is recommended for further studies to unravel further properties of the set of non deranged permutations coordinates in the Euclidean space. The minimum distance between two non deranged permutation coordinates in the Euclidean space should be studied.

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CONFLICT OF INTEREST

I declare that no conflict of interest exist.

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