Stability Analysis of Periodic Solutions of a Certain Ultra-hyperbolic Boundary Value Problem

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Abstract- ***In this paper, stability analysis of periodic solutions of ultra-hyperbolic boundary value problem was investigated using eigenvalue approach. Periodic solutions for the ultra-hyperbolic equation was obtained using the method of separation of variables and the proof of periodicity was conducted for the independent variables. Test for periodicity was conducted for the independent variables and the results show that the system obey periodic solutions. Stability results for the ultra-hyperbolic equation was obtained by converting the system of equations into first order equivalent system and further obtained the matrix for the given system. The result obtained show that the equilibria points were not asymptotically stable. Eight eigenvalues were obtained in which four of them are parameter dependent and the other four eigenvalues are equal to one. Furthermore, numerical simulations were used to demonstrate the behavior of the system for different values of the parameters extending known results in literature. Application of our results can be seen in inverted pendulum where amplitude of oscillation grows exponentially or without bound.***

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I. INTRODUCTION

In this paper, we consider the stability analysis of periodic solutions of ultra-hyperbolic boundary value problem of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=0\ , \qquad (1.1)$$

With the following boundary and initial conditions

$u(0,y,z,t)=0\ \ u(L_x,y,z,t)=0\ \ 0\le x\le L_x$
$t\ge 0$

$u(x,0,z,t)=0\ \ u(x,L_y,z,t)=0\ \ 0\le y\le L_y$

$u(x,y,0,t)=0\ \ u(x,y,L_z,t)=0\ \ 0\le z\le L_z$

$u(x,y,z,0)=f(x,y,z)\ \ \frac{\partial u}{\partial t}(x,y,z,0)=g(x,y,z)$

L_x represent the length of the domain in the x direction

L_y represent the length of the domain in the y direction

L_Z represent the length of the domain in the z direction

u is the unknown scalar function and x,y,z,t are the independent variables. Equation (1.1) is a time-dependent wave model that model how disturbance evolve over time in a medium that supports oscillations or signals.

Ultra-hyperbolic equation is a partial differential equation for an unknown scalar function u of $2n$ variables $x_1,\dots,x_n,\ y_1,\dots,y_n$ of the form

$$\frac{\partial^2 u}{\partial x_1^2}+\dots+\frac{\partial^2 u}{\partial x_n^2}-\frac{\partial^2 u}{\partial y_1^2}-\dots-\frac{\partial^2 u}{\partial y_n^2}=0 \qquad (1.2)$$

see [1]. The equation resembles the classical wave equation which has led to number of developments due to its characteristics [2]. The applications of ultra-hyperbolic equation can be seen in the modelling of space time [3], designing of harbours and dams, earthquake analysis (electrodynamic wave propagation) and design of antennas (electromagnetic wave propagation) [4]. The equation can also be used in the study of symmetric spaces and elliptic differential operators [5]. Due to the characteristics of the ultra-hyperbolic equation, many authors have studied ultra-hyperbolic equation producing sound results. For instance see [6], [7], [8], [9] and there references therein.

Stability is a qualitative property of a differential equation that describes the behavior of trajectories under small displacement of initial conditions. The key idea in stability is that the qualitative behavior of an equilibrium point can be analyzed using the linearization of the system near the equilibrium point [10]. Some authors have worked on stability analysis of ultra-hyperbolic equation using different methods. Golgeleyen and Yamamoto (2014) considered stability of inverse problem for ultra-hyperbolic equation using Carleman estimate by some lateral boundary data and concluded by proving holder estimate which are global and local. Golgeleyen and

Kaytmaz (2020) investigated conditional stability for a Cauchy problem for the ultra-hyperbolic Schrodinger equation. They first establish a local Carleman estimate for the model which arises in some theories of modern Physics and concluded by proving a holder stability estimate for the Cauchy problem. However to the best of our knowledge, the stability analysis of ultrahyperbolic equation of four independent variables that is time dependent using eigenvalue approach is still yet not covered. It is also essential to continue to study stability analysis of (1.1) because of its wide area of applications. The use of Eigenvalue approach in this case to study the behavior of the four independent variables around the equilibrium points.

Motivated by the above literature, the objective of this paper is to investigate stability analysis of periodic solutions of (1.1) using eigenvalue approach. The periodic solution and test for periodicity will be established using separation of variable and Fourier series. In section 2, we introduce some theorems which will help us to obtain our main results. The analysis will be shown in section 3 while the paper is concluded in section 5 with a numerical simulations in section 4 to demonstrate the behavior of the system.

II. PERLIMINARIES

Definition 2.1 (Separation of variable method) [13]

Separation of variables is a method of solving ordinary and partial differentiation. For a partial differential equation in a function $\phi(x, y, \ldots)$ and variables $x, y, \ldots$, separation of variables can be applied by making a substitution of the form

$$\phi(x, y, \ldots) = X(x)Y(y)\ldots, \qquad (2.1)$$

breaking the resulting equation into a set of independent ordinary differential equations, solving these for $X(x), Y(y)$, . . . , and then plugging them back into the original equation. Separation of variables was used by L'Hospital in 1970. It is useful in solving equations arising in Mathematical Physics such as Laplace's equation, the Helmholtz differential equation and the Schrodinger equation

Definition 2.2 (Periodic Solution)
A solution $u(x, y, z, t)$ to a partial differential equation with four independent variables is said to be periodic in one or more variables if there exist $T > 0$, $T_x > 0, T_y > 0$ and $T_z > 0$ such that

$$u(x, y, z, t + T) = u(x, y, z, t)$$

$$u(x + T_x, y, z, t) = u(x, y, z, t)$$

$$u\left(x, y + T_y, z, t\right) = u(x, y, z, t)$$

$$u(x, y, z + T_z, t) = u(x, y, z, t)$$

for $(x, y, z, t) \in \mathbb{R}$. T, T_x, T_y and T_z are the periods

Definition 2.3 (Periodic Solution)
A periodic solution is a solution of the equation with the property that there exist a positive real number $T > 0$ such that $x(t + T) = x(t)$ for all $t \in \mathbb{R}$
Definition 2.4 (Fourier series)

This is a technique used to convert a complex wave function into a single sine or cosine function. Let $f(x)$ be a function in the interval $[-L, L]$, then Fourier series formula is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$,

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad n > 0,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n > 0$ are Fourier coefficients of f.

Definition 2.5 For a PDE involving an unknown function u of four independent variables, say x_1, x_2, x_3, x_4, a solution is a function $u(x_1, x_2, x_3, x_4)$ defined on a specific domain that makes the equation a true statement over that domain.

Definition 2.6 (Eigenvalue Stability):
The concept uses the eigenvalue to access the stability of a given system. The stability behavior depends on the existence of real, imaginary components of the system. To investigate the stability of the equilibrium point using the eigenvalue approach, the following steps are necessary.
(i) Linearize the second order differential equation to obtain a matrix say A.
(ii) Determine the equilibrium point
(iii) Determine the eigenvalue of the equilibrium point
(iv) Determine the stability based on the sign of the eigenvalue

Table 3.8.1 Stability Corresponding to each type of eigenvalue

Eigenvalue Type	Stability
$All\ real\ and\ +$	$Unstable$
$All\ real\ and\ -$	$Stable$
$Mixed + and - real$	$Unstable$
$a+bi$	$Unstable$
$-a+bi$	$Stable$
$0+bi$	$Unstable$

Theorem 2.7 (Superposition Principle) [14]

The principle of superposition state that if u_1 and u_2 are solutions of a linear PDEs then the linear combination $\alpha u_1 + \beta u_2$ is a solution where α and β are arbitrary constants.

Consider the ultra-hyperbolic equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=0 \tag{2.2}$$

Suppose that $u = u_1(x,y,z,t$) and $u = u_2(x,y,z,t)$ are solutions to equation (2.2). We consider a linear combination of u_1 and u_2 by letting $u = \alpha u_1 + \beta u_2$ where α and β are constants. The principle of superposition states that u is a solution of equation (2.2). To prove this, we compute

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2(\alpha u_1+\beta u_2)}{\partial x^2}+\frac{\partial^2(\alpha u_1+\beta u_2)}{\partial y^2}-\frac{\partial^2(\alpha u_1+\beta u_2)}{\partial z^2}-\frac{\partial^2(\alpha u_1+\beta u_2)}{\partial t^2} \tag{2.3}$$

$$=\frac{\partial^2 \alpha u_1}{\partial x^2}+\frac{\partial^2 \beta u_2}{\partial x^2}+\frac{\partial^2 \alpha u_1}{\partial y^2}+\frac{\partial^2 \beta u_2}{\partial y^2}-\frac{\partial^2 \alpha u_1}{\partial z^2}-\frac{\partial^2 \beta u_2}{\partial z^2}-\frac{\partial^2 \alpha u_1}{\partial t^2}-\frac{\partial^2 \beta u_2}{\partial t^2} \tag{2.4}$$

$$=\alpha\frac{\partial^2 u_1}{\partial x^2}+\beta\frac{\partial^2 u_2}{\partial x^2}+\alpha\frac{\partial^2 u_1}{\partial y^2}+\beta\frac{\partial^2 u_2}{\partial y^2}-\alpha\frac{\partial^2 u_1}{\partial z^2}-\beta\frac{\partial^2 u_2}{\partial z^2}-\alpha\frac{\partial^2 u_1}{\partial t^2}-\beta\frac{\partial^2 u_2}{\partial t^2} \tag{2.5}$$

$$=\alpha\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}-\frac{\partial^2 u_1}{\partial z^2}-\frac{\partial^2 u_1}{\partial t^2}\right)+\beta(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}-\frac{\partial^2 u_2}{\partial z^2}-\frac{\partial^2 u_2}{\partial t^2}) \tag{2.6}$$

$$=\alpha\times 0+\beta\times 0=0$$

Since u_1 and u_2 were assumed to be solution of equation (2.2). We have therefore shown that any linear combination of solution to homogenous ultra-hyperbolic equation is also a solution.

III. MAIN RESULTS

3.1 Periodic Solutions of Ultra-hyperbolic equation

We consider the partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=0\ , \tag{3.1}$$

with the following boundary and initial conditions

$u(0,y,z,t)=0\ \ u(L_x,y,z,t)=0\ 0\le x\le L_x\quad t\ge 0$

$u(x,0,z,t)=0\ \ u\big(x,L_y,z,t\big)=0\ 0\le y\le L_y$

$u(x,y,0,t)=0\ \ u(x,y,L_z,t)=0\ 0\le z\le L_z$

$u(x,y,z,0)=f(x,y,z)\ \ \frac{\partial u}{\partial t}(x,y,z,0)=g(x,y,z)$

To obtain periodic solution of (3.1) we assume a solution of the form

$$u(x,y,z,t)=X(x)Y(x)Z(z)T(t) \tag{3.2}$$

Substituting equation (3.1) in equation (3.2) gives

$$\frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial x^2}+\frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial y^2}-\frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial z^2}-\frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial t^2}=0 \tag{3.3}$$

$$YZT\frac{d^2X}{dx^2}+XZT\frac{d^2Y}{dy^2}-XYT\frac{d^2Z}{dz^2}-XYZ\frac{d^2T}{dt^2}=0 \tag{3.4}$$

Dividing by $XYZT$ we have

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}-\frac{1}{Z}\frac{d^2Z}{dz^2}-\frac{1}{T}\frac{d^2T}{dt^2}=0 \tag{3.5}$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{1}{T}\frac{d^2T}{dt^2} \tag{3.6}$$

Since the LHS is a function of x and y and RHS is a function of z and t, we set equation (3.6) equal to k where k is the separation constant.

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{1}{T}\frac{d^2T}{dt^2}=k \tag{3.7}$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=k \text{ and } \frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{1}{T}\frac{d^2T}{dt^2}=k \tag{3.8}$$

Considering $\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=k$ we have

$$\frac{1}{X}\frac{d^2X}{dx^2}=k-\frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda \tag{3.9}$$

Where λ is another separation constant. Hence we have

$\frac{1}{X}\frac{d^2X}{dx^2}=\lambda$ which gives $X''(x)-\lambda X(x)=0$. Simplifying $k-\frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda$ gives $Y''(y)-vY(y)=0$ where $v=k-\lambda$. For $\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{1}{T}\frac{d^2T}{dt^2}=k$ we have $\frac{1}{Z}\frac{d^2Z}{dz^2}=k-\frac{1}{T}\frac{d^2T}{dt^2}=\tau$ where τ is another constant. Simplifying further gives $Z''(z)-\tau Z(z)=0$ and $T''(t)-wT(t)=0$ where $w=k-\tau$. The system of equations is given by

$$X''(x)-\lambda X(x)=0 \tag{3.10}$$

$$Y''(y)-vY(y)=0 \tag{3.11}$$

$$Z''(z)-\tau Z(z)=0 \tag{3.12}$$

$$T''(t)-wT(t)=0 \tag{3.13}$$

Case 1: Considering for $\lambda>0$, $v>0, \tau>0$ and $w>0$ we have

$$X(x)=c_1e^{\sqrt{\lambda}x}+c_2e^{-\sqrt{\lambda}x} \tag{3.14}$$

$$Y(y)=c_3e^{\sqrt{v}y}+c_4e^{-\sqrt{v}y} \tag{3.15}$$

$$Z(z)=c_5e^{\sqrt{\tau}z}+c_6e^{-\sqrt{\tau}z} \tag{3.16}$$

$$T(t)=c_7e^{\sqrt{w}t}+c_8e^{-\sqrt{w}t} \tag{3.17}$$

Where $c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8$ are constants. Hence the general solution for case 1 is given by

$$u(x,y,z,t) = (c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x})(c_3e^{\sqrt{v}y} + c_4e^{-\sqrt{v}y}\ )\ (c_5e^{\sqrt{\tau}z} + c_6e^{-\sqrt{\tau}z})\ (c_7e^{\sqrt{w}t} + c_8e^{-\sqrt{w}t})$$

Applying the boundary conditions we have

$$u(0.y,z,t) = 0 = (c_1 + c_2)\ (c_3e^{\sqrt{v}y} + c_4e^{-\sqrt{v}y}\ )\ (c_5e^{\sqrt{\tau}z} + c_6e^{-\sqrt{\tau}z})\ (c_7e^{\sqrt{w}t} + c_8e^{-\sqrt{w}t})$$

Which implies that

$$c_1 + c_2 = 0 \tag{3.18}$$

Also $u(L_x, y, z, t) = 0$ gives

$$c_1e^{\sqrt{\lambda}L_x} + c_2e^{-\sqrt{\lambda}L_x} = 0 \tag{3.19}$$

$$u(x,0,z,t) = 0 = c_3 + c_4(c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x})(c_5e^{\sqrt{\tau}z} + c_6e^{-\sqrt{\tau}z})(c_7e^{\sqrt{w}t} + c_8e^{-\sqrt{w}t})$$

implies that

$$c_3 + c_4 = 0 \tag{3.20}$$

And $u\left(x, L_y, z, t\right) = 0$ gives

$$c_3e^{\sqrt{v}L_y} + c_2e^{-\sqrt{v}L_y} = 0 \tag{3.21}$$

$$u(x,y,0,t) = 0 = (c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x})(c_3e^{\sqrt{v}y} + c_4e^{-\sqrt{v}y})(c_5 + c_6)(c_7e^{\sqrt{w}t} + c_8e^{-\sqrt{w}t})$$

implies that

$$c_5 + c_6 = 0 \tag{3.22}$$

$$u(x,y,L_z,t) = (c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x})(c_3e^{\sqrt{v}y} + c_4e^{-\sqrt{v}y}\ )(c_5e^{\sqrt{\tau}L_z} + c_6e^{-\sqrt{\tau}L_z})$$

$$(c_7e^{\sqrt{w}t} + c_8e^{-\sqrt{w}t})$$

$u(x, y, L_z, t) = 0$ gives

$$(c_5e^{\sqrt{\tau}L_z} + c_6e^{-\sqrt{\tau}L_z}) = 0 \tag{3.23}$$

Equations (3.18) and (3.19) possess a non-trivial solution if and only if

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}L_x} & e^{-\sqrt{\lambda}L_x} \end{vmatrix} = 0$$

$$e^{-\sqrt{\lambda}L_x} - e^{\sqrt{\lambda}L_x} = 0 \tag{3.24}$$

Dividing equation (4.1.31) by $e^{-\sqrt{\lambda}L_x}$ we have

$$1 - e^{2\sqrt{\lambda}L_x} = 0 \tag{3.25}$$

Equation (3.25) implies that $e^{2\sqrt{\lambda}L_x} = 1$. Further simplification gives $\sqrt{\lambda}L_x = 0$. This implies that $\sqrt{\lambda} = 0$ since $L_x \neq 0$. The same result goes for equations (3.20) and (3.21). This results is against the assumption for Case 1. Hence the solution is not acceptable.

Case II: for $\lambda = 0, v = 0, \tau = 0$ and $w = 0$ we have

$$\frac{d^2X}{dx^2} = 0 \tag{3.26}$$

$$\frac{d^2Y}{dy^2} = 0 \tag{3.27}$$

$$\frac{d^2Z}{dz^2} = 0 \tag{3.28}$$

$$\frac{d^2T}{dt^2} = 0 \tag{3.29}$$

Solutions for equations (3.26), (3.27), (3.28) and (3.29) is given by

$$X = Ax + B \tag{3.30}$$

$$Y = Cy + D \tag{3.31}$$

$$Z = Ez + F \tag{3.32}$$

$$T = Gt + H \tag{3.33}$$

Where B, D, F and H are constants. Therefore the required solution is

$$u(x,y,z,t) = (Ax + B)(Cy + D)(Ez + F)(Gt + H) \tag{3.34}$$

Using the boundary conditions we have

$u(0.y,z,t) = 0 = B(Cy + D)(Ez + F)(Gt + H)$. This implies that $B = 0$

$u(L_x,y,z,t) = 0 = AL_x(Cy + D)(Ez + F)(Gt + H)$. This implies that $A = 0$

$u(x,0,z,t) = 0 = D(Ax + B)(Ez + F)(Gt + H)$. This implies that $D = 0$

$u(x,L_y,z,t) = 0 = CL_y(Ax + B)(Ez + F)(Gt + H)$. This implies that $C = 0$

$u(x,y,0,t) = 0 = F(Ax + B)(Cy + D)(Gt + H)$. This implies that $F = 0$

$u(x,y,L_z,t) = 0 = EL_z(Ax + B)(Cy + D)(Gt + H)$. This implies $E = 0$

Hence only trivial solution is possible.

Case III: for $\lambda < 0: \lambda = -\alpha^2, \alpha > 0, v < 0: v = -\beta^2, \beta > 0, \tau < 0: \tau = -\gamma^2, \gamma > 0$ and $w < 0: w = -\eta^2, \eta > 0$. The differential equations are

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \tag{3.35}$$

$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0 \tag{3.36}$$

$$\frac{d^2Z}{dz^2} + \gamma^2 Z = 0 \tag{3.37}$$

$$\frac{d^2T}{dt^2} + \eta^2 T = 0 \tag{3.38}$$

Auxiliary equation for equation (3.35) is given by

$$m^2 + \alpha^2 = 0 \tag{3.39}$$

$m = \pm\sqrt{-\alpha^2},\ m = \pm i\alpha$. Hence the solution of equation of (3.35) is given by

$$X(x) = c_1 cos\alpha x + c_2 sin\alpha x \tag{3.40}$$

Similarly for equations (3.36), (3.37) and (3.38), the solutions are

$$Y(y) = c_3 cos\beta y + c_4 sin\beta y \tag{3.41}$$

$$Z(z) = c_5 cos\gamma z + c_6 sin\gamma z \tag{3.42}$$

$$T(x) = c_7 cos\eta t + c_8 sin\eta t \qquad (3.43)$$

Hence the general solution is given by

$$u(x, y, z, t) = (c_1 cos\alpha x + c_2 sin\alpha x)(c_3 cos\beta y + c_4 sin\beta y)(c_5 cos\gamma z + c_6 sin\gamma z)$$
$$(c_7 cos\eta t + c_8 sin\eta t) \qquad (3.44)$$

Using the boundary conditions we have that for

$u(0, y, z, t) = 0, \ \ c_1 = 0$

$u(L_x, y, z, t) = 0, \qquad sin\alpha L_x = 0$

$u(x, 0, z, t) = 0, \ \ c_3 = 0$

$u\left(x, L_y, z, t\right) = 0, \qquad sin\beta L_y = 0$

$u(x, y, 0, t) = 0, \qquad c_5 = 0$

$u(x, y, L_z, t) = 0, \qquad sin\gamma L_z = 0$

For $sin\alpha L_x = 0, \ \alpha_n = \frac{n\pi}{L_x}, \ n = 1,2, \dots,$ are the eigenvalues. The solution is

$$u_n(x, t) = sin\frac{n\pi x}{L_x}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right), \quad n = 1,2, \ \dots, \qquad (3.45)$$

For $sin\beta L_y = 0, \ \beta_m = \frac{m\pi}{L_y}, \ m = 1,2, \dots,$ are the eigenvalues. The solution is

$$u_m(y, t) = sin\frac{m\pi y}{L_y}\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right), \quad m = 1,2, \ \dots, \qquad (3.46)$$

For $sin\gamma L_z = 0, \ \gamma_g = \frac{g\pi}{L_z}, g = 1,2, \dots,$ are the eigenvalues. The solution is

$$u_g(z, t) = sin\frac{g\pi z}{L_z}\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right), \ g = 1,2, \ \dots, \qquad (3.47)$$

Combining equations (3.45), (3.46) and (3.47) we have

$$u_p(x, y, z, t) = sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)$$
$$\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) \qquad (3.48)$$

Where $p = nmg$ and $A_n, B_n, C_m, D_m, E_g, F_g$ are constant coefficients

Using the superposition principle we have

$$u(x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$
$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) \qquad (3.49)$$

Applying the initial conditions gives

$$u(x, y, z, 0) = f(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}(A_n \ )(C_m)(E_g)$$

$$u(x, y, z, 0) = f(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\alpha=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} J_q \qquad (3.50)$$

Where $J_q = (A_n)(C_m)(E_g)$

J_q is the half range Fourier sine series for A_n, C_m and E_g where

$$A_n = \frac{2}{L_x}\int_0^{L_x} f(x,y,z) sin\frac{n\pi x}{L_x}dx \qquad (3.51)$$

$$C_m = \frac{2}{L_y}\int_0^{L_y} f(x,y,z) sin\frac{m\pi y}{L_y}dy \qquad (3.52)$$

$$E_g = \frac{2}{L_z}\int_0^{L_z} f(x,y,z) sin\frac{g\pi z}{L_z}dz \qquad (3.53)$$

Combining equations (3.51), (3.52) and (3.53) we have

$$G_w = \frac{8}{L_xL_yL_z}\int_0^{L_x}\int_0^{L_y}\int_0^{L_z} f(x,y,z) sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y}\ sin\frac{g\pi z}{L_z}dxdydz \qquad (3.54)$$

$$\frac{\partial u}{\partial t}(x,y,z,t) = \left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)\left(\frac{-m\pi}{L_y}C_m sin\frac{m\pi t}{L_y} + \frac{m\pi}{L_y}D_m cos\frac{m\pi t}{L_y}\right)$$

$$\left(\frac{-g\pi}{L_z}E_g sin\frac{g\pi t}{L_z} + \frac{g\pi}{L_z}F_g cos\frac{g\pi t}{L_z}\right) + \left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(\frac{-g\pi}{L_z}E_g sin\frac{g\pi t}{L_z} + \frac{g\pi}{L_z}F_g cos\frac{g\pi t}{L_z}\right)$$

$$\left(\frac{-n\pi}{L_x}A_n sin\frac{n\pi t}{L_x} + \frac{n\pi}{L_x}B_n cos\frac{n\pi t}{L_x}\right) + \left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right)\left(\frac{-n\pi}{L_x}A_n sin\frac{n\pi t}{L_x} + \frac{n\pi}{L_x}B_n cos\frac{n\pi t}{L_x}\right)$$

$$\left(\frac{-m\pi}{L_y}C_m sin\frac{m\pi t}{L_y} + \frac{m\pi}{L_y}D_m cos\frac{m\pi t}{L_y}\right) \qquad (3.55)$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = A_nD_mF_g\frac{m\pi}{L_y}\frac{g\pi}{L_z} + C_mF_gB_n\frac{g\pi}{L_z}\frac{n\pi}{L_x} + E_gB_nD_m\frac{n\pi}{L_x}\frac{m\pi}{L_y} \qquad (3.56)$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = g(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\alpha=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$$

$$\left(A_nD_mF_g\frac{m\pi}{L_y}\frac{g\pi}{L_z} + C_mF_gB_n\frac{g\pi}{L_z}\frac{n\pi}{L_x} + E_gB_nD_m\frac{n\pi}{L_x}\frac{m\pi}{L_y}\right) \qquad (3.57)$$

Hence the required solution is equation (3.49). Equation (3.49) is periodic in nature since the solution contain sinusoidal functions. The constant term and periodic term in (3.49) shows that the solution is periodic which is unstable in the sense that all solution starting sufficiently close to it depart from it with increasing time.

3.2 Test for Periodicity of Ultrahyperbolic Equation using Fourier Series Method

We first establish the period for each variable. The formula for period of a periodic function given by $\frac{2\pi}{n}$ where n is the coefficient of x and 2π is the fundamental period.

The term $sin\frac{n\pi x}{L_x}$ is periodic in x with period $T_x = \frac{2\pi}{\frac{n\pi}{L_x}} = \frac{2L_x}{n}$

The term $cos\frac{n\pi t}{L_x}$ is periodic in t with period $T = \frac{2L_x}{n}$

The term $sin\frac{m\pi y}{L_y}$ is periodic in y with period $T_y = \frac{2L_y}{m}$

The term $sin\frac{g\pi z}{L_z}$ is periodic in z with period $T_z = \frac{2L_z}{g}$

Next is to establish Fourier Series coefficient for each variable.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L_x}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L_x}\right) \qquad (3.58)$$

For $f(x) = sin\frac{n\pi x}{L_x}$, Let $\frac{n\pi}{L_x} = c$, we have

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{L} sincxdx \qquad (3.59)$$

$= \frac{1}{L}\left[\frac{-coscx}{c}\right]_{-L}^{L} = 0$ for $c \neq 0$ (3.60)

$a_n = \frac{1}{L}\int_{-L}^{L} sincxcoscxdx$ (3.61)

$\sin(c+c)\,x = sincxcoscx + coscxsincx$

$sin2cx = 2sincxcoscx$. Hence $sincxcoscx = \frac{sin2cx}{2}$

$\frac{1}{L}\int_{-L}^{L}\frac{sin2cx}{2}dx = \frac{1}{2L}\left[\frac{-cos2cx}{2c}\right]_{-L}^{L} = 0$ (3.62)

$b_n = \frac{1}{L}\int_{-L}^{L} sincxsincxdx = \frac{1}{L} \qquad \int_{-L}^{L} sin^2\, cxdx$ (3.63)

$= \frac{1}{L}\int_{-L}^{L}\frac{1-cos2cx}{2}dx = \frac{1}{L}\left[\frac{x}{2} - \frac{sin2cx}{4c}\right]_{-L}^{L}$ (3.64)

$= 1 - \frac{sin2cL}{4c} + \frac{sin2cL}{4c} = 1$ (3.65)

Therefore the Fourier series for variable x in given by

$f(x) = \sum_{n=1}^{\infty} sin\frac{n\pi x}{L_x} \qquad n = 1,2,\ldots$ (3.66)

Similarly for the function $f(y) = sin\frac{m\pi y}{L_y}$ .Then the Fourier series is given by

$f(y) = \sum_{m=1}^{\infty} sin\frac{m\pi y}{L_y} \quad m = 1,2,\ldots$ (3.67)

For the function $f(z) = sin\frac{g\pi z}{L_z}$.. Then the Fourier series is given by

$f(z) = \sum_{\alpha=1}^{\infty} sin\frac{g\pi z}{L_z} \quad g = 1,2,\ldots$ (3.68)

For the function $f(t) = cos\frac{n\pi t}{L_x}$, the Fourier series is gven by

$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi t}{L_x}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi t}{L_x}\right)$ (3.69)

Let $\frac{n\pi}{L_x} = c$, then the coefficients are given by

$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt = \frac{1}{L}\int_{-L}^{L} cosctdt$ (3.70)

$= \frac{1}{L}\left[\frac{sinct}{c}\right]_{-L}^{L} = \frac{1}{L}\left[\frac{sincL}{c} - \frac{sinc(-L)}{c}\right] = 0$ (3.71)

This is because $sincL = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} cosctcosctdt = \frac{1}{L}\int_{-L}^{L} cos^2ctdt$ (3.72)

$= \frac{1}{L}\int_{-L}^{L}\frac{cos2ct+1}{2}dt$ since $cos2A = 2cos^2A - 1$

$a_n = \frac{1}{L}\int_{-L}^{L}\frac{cos2ct}{2}dt + \frac{1}{L}\int_{-L}^{L}\frac{1}{2}dt$ (3.73)

$= \frac{1}{L}\left[\frac{sin2ct}{4c}\right]_{-L}^{L} + \frac{1}{L}\left[\frac{t}{2}\right]_{-L}^{L}$ (3.74)

$= \frac{1}{L}\left[\frac{sin2cL}{4c} - \frac{sin2c(-L)}{4c}\right] + \frac{1}{L}\left[\frac{L}{2} + \frac{L}{2}\right] = 1$ (3.75)

$$b_n = \frac{1}{L}\int_{-L}^{L} cosctsinctdt \qquad (3.76)$$

$$sin2ct = sinctcosct + cosctsinct$$

$$sin2ct = 2sinctcosct$$

$$cosctsinct = \frac{sin2ct}{2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \frac{sin2ct}{2}dt = \frac{1}{2L}\left[\frac{-cos2ct}{2c}\right]_{-L}^{L} \qquad (3.77)$$

$$= -\frac{cos2cL}{4Lc} + \frac{cos2c(-L)}{4Lc} = 0 \qquad (3.78)$$

Therefore the Fourier series is given by

$$f(t) = \sum_{n=1}^{\infty} cos\frac{n\pi t}{L_x} \quad \text{for } n = 1,2.3,\ \ldots \qquad (3.79)$$

Since each variable has a Fourier series representation at a particular frequency and the Fourier coefficient are not all zeros, then the solution of ultrahyperbolic equation exhibits periodicity at that frequency.

To show that the solution of ultra-hyperbolic equation for four independent is periodic we established that

1. $u(x + T_x, y, z, t) = u(x, y, z, t)$

$$u(x + T_x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$

$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) \qquad (3.80)$$

Considering $sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$ we have

$$sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = \sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \qquad (3.81)$$

Note that $\sin(A + B) = sinAcosB + cosAsinB$. Let $\frac{n\pi x}{L_x} = A$ and $\frac{n\pi T_x}{L_x} = B$

$$sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) = sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x} \qquad (3.82)$$

Equation (3.81) becomes

$$\sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = \left(sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$$

$$= sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x}\ sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \qquad (3.83)$$

Since $T_x = \frac{2L_x}{n}$, $cos\frac{n\pi T_x}{L_x} = \cos 2\pi = 1$. Also $sin\frac{n\pi T_x}{L_x} = \sin 2\pi = 0$. Then we have

$$\sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \qquad (3.84)$$

$$u(x + T_x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$

$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right)$$

$$= u(x, y, z, t) \qquad (3.85)$$

The same results is true for y, z and t. Hence we conclude that the solution of ultra-hyperbolic equation is periodic.

3.2 Stability analysis of Ultra-hyperbolic equation

Consider the ultra-hyperbolic equation of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=0 \qquad (3.86)$$

We assume a solution of the form

$$u(x,y,z,t)=X(x)Y(y)Z(z)T(t) \qquad (3.87)$$

$u_x = X'YZT,\ u_{xx} = X''YZT,\ u_y = XY'ZT,\ u_{yy} = XY''ZT,\ u_z = XYZ'T,\ u_{zz} = XYZ''T,$

$u_t = XYZT',\ u_{tt} = XYZT''$

Substituting in equation (3.86) gives

$$\frac{X''YZT}{XYZT}+\frac{XY''ZT}{XYZT}-\frac{XYZ''T}{XYZT}-\frac{XYZT''}{XYZT}=0 \qquad (3.88)$$

$$\frac{X''}{X}+\frac{Y''}{Y}-\frac{Z''}{Z}-\frac{T''}{T}=0 \qquad (3.89)$$

$$\frac{X''}{X}+\frac{Y''}{Y}=\frac{Z''}{Z}+\frac{T''}{T} \qquad (3.90)$$

Setting equation (3.90) equal to a constant gives

$$\frac{X''}{X}+\frac{Y''}{Y}=\frac{Z''}{Z}+\frac{T''}{T}=k \qquad (3.91)$$

Consider $\frac{X''}{X}+\frac{Y''}{Y}=k$ we have $\frac{X''}{X}=k-\frac{Y''}{Y}=e$ where k, e are constants. $\frac{X''}{X}=e$ implies $X''-eX=0$. $k-\frac{Y''}{Y}=e$ implies $Y''-\mu Y=0$ where $\mu=k-e$. For $\frac{Z''}{Z}+\frac{T''}{T}=k$ we have $\frac{Z''}{Z}=k-\frac{T''}{T}=\sigma$ where σ is another constant. $\frac{Z''}{Z}=\sigma$ implies $Z''-\sigma Z=0$. $k-\frac{T''}{T}=\sigma$ implies $T''-vT=0$ where $v=k-\sigma$. Hence the required system of equations is given by

$$X''-eX=0 \qquad (3.92)$$

$$Y''-\mu Y=0 \qquad (3.93)$$

$$Z''-\sigma Z=0 \qquad (3.94)$$

$$T''-vT=0 \qquad (3.95)$$

Converting the system of equations into first order equivalent system we have

Let $X'=u,\ u'=X''=eX,\ Y'=w, w'=Y''=\mu Y,\ Z'=c, c'=Z''=\sigma Z,$

$T'=d,\ d'=T''=vT$ Hence the first order system is given by

$$\left.\begin{aligned} X'&=u\\ u'&=eX\\ Y'&=w\\ w'&=\mu Y\\ Z'&=c\\ c'&=\sigma Z\\ T'&=d\\ d'&=vT \end{aligned}\right\} \qquad (3.96)$$

Representing the above first order system in the matrix form $x'=Ax$ we have

$$\begin{bmatrix} X' \\ u' \\ Y' \\ w' \\ Z' \\ c' \\ T' \\ d' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{bmatrix} \begin{bmatrix} u \\ X \\ w \\ Y \\ c \\ Z \\ d \\ T \end{bmatrix} \qquad (3.97)$$

Where $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{bmatrix}$, $x' = [X' \quad u' \quad Y' \quad w' \quad Z' \quad c' \quad T' \quad d']^T$

and $x = [u \quad X \quad w \quad Y \quad c \quad Z \quad d \quad T]^T$

Since the matrix is 8×8 diagonal matrix, the eigenvalues are obtained using $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v-\lambda \end{vmatrix} = 0 \qquad (3.98)$$

$$(1-\lambda)(e-\lambda)(1-\lambda)(\mu-\lambda)(1-\lambda)(\sigma-\lambda)(1-\lambda)(v-\lambda) = 0 \qquad (3.99)$$

Solving equation (3.99) gives

$\lambda = 1(4\, times)$, $\lambda = e$, $\lambda = \mu$, $\lambda = \sigma$, and $\lambda = v$

Since all the eigenvalues are non-negative, the equilibrium point is not asymptotically stable. The stability results is local stability which describes the behavior of the system around the equilibrium points.

IV. NUMERICAL SIMULATIONS OF ULTRA-HYPERBOLIC EQUATION

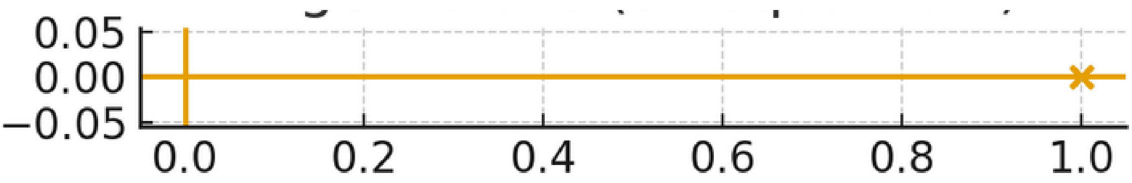

Figure 1. Phase portrait whose eigenvalues are all equal to one.

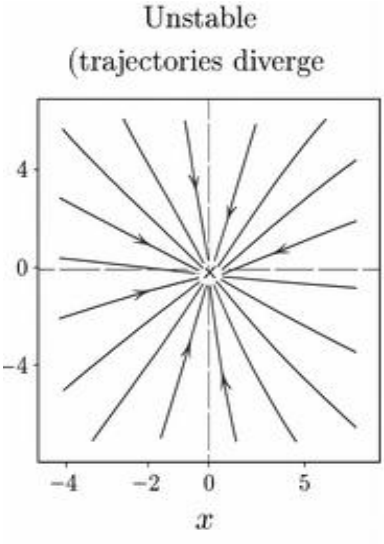

Figure 2. Phase portrait describing unstable equilibrium points.

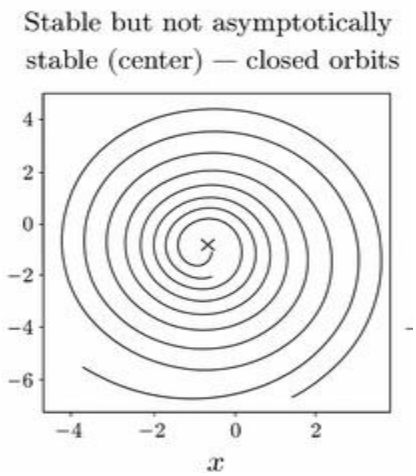

Figure 3. Phase portrait showing stable system

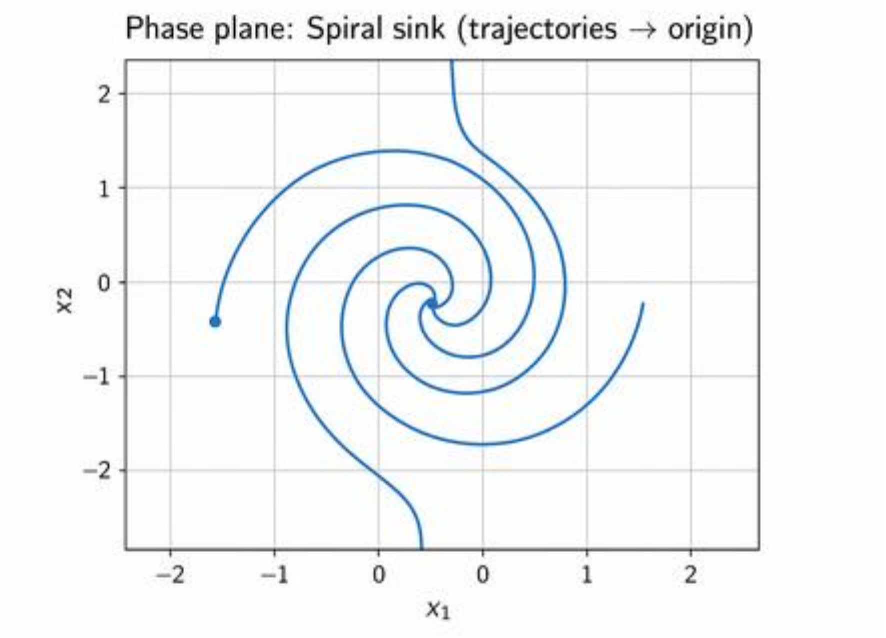

Figure 4. Phase portrait describing asymptotic stability.

V. CONCLUSIONS

From our results, eigenvalue approach is very effective in investigating stability analysis of ultra-hyperbolic equation in which some of the eigenvalues parameter dependent. The advantages of this method are that it provides insight into the system behavior and provide a clear stability criterion. Hence, we conclude that the stability analysis of ultra-hyperbolic equation is unstable and also depend on the parameters. Application of our results can be seen in inverted pendulum where a little displacement from the equilibrium position leads to an increase oscillation thereby causing it to fall over. The choice of different values of the parameters will demonstrate other stability results thereby providing insight into behavior of the system for ultra-hyperbolic equation. The numerical behavior are explained as follows:

Figure 1 describes the behavior of the system when the four eigenvalues satisfy $Re(\lambda) = 1 > 0$. This represent an unstable system in which the system cannot return to equilibrium. In this case, the inverted pendulum will fall over on any little displacement from the equilibrium position.

Figure 2 describes an unstable system. This behavior is obtained as a result of varying the parameters. In this case, the inverted pendulum will fall due to small change in the initial condition.

Figure 3 describes a stable system that is not asymptotically stable. In this case, the system return to equilibrium state after a disturbance. Example of this behavior can be seen in shock absorbers in vehicles.

Figure 4 describes an asymptotically stable system. This occurs when the eigenvalues have negative real part. In this case, any little displacement from the equilibrium position will return the trajectory back to the origin. Example of this behavior can be seen in a pendulum.

Competing Interests

The authors declared that they have no competing interests.

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